Nonlinear Dispersive Models

Well-posedness and Unique Continuation Property

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To memory of my parents

Acknowledgments

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Contents

Notation

Here we give some notations that we are going to use throughout this work.

- $\mathbb{N}$ - set of natural numbers
- $\mathbb{R}$ - set of real numbers
- $\mathbb{Z}$ - set of integers
- $\mathbb{C}$ - set of complex numbers
- $\partial_x^k u$ or $u_{x\cdots x}$ or $\frac{\partial^k u}{\partial x^k}$ - partial derivative of u w.r.t. variable x of order k
- $\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$ - Fourier transform of f
- $f^{\vee}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi$ - inverse Fourier transform of f
- $D_x^s f := (-\partial_x^2)^{s/2} f = [|\cdot|^s \hat{f}(\cdot)]^{\vee}$ - Riesz potential of order $-s$.
- $\mathcal{S}(\mathbb{R}^n)$ - Schwartz space on $\mathbb{R}^n$
- $C([0,T];X)$ - space of continuous functions from $[0,T]$ into X
- $\|f\|_s := (\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi)^{1/2}$
- $H^s(\mathbb{R}) := H^s$ - Sobolev space of order s with norm $\|f\|_s$
- $L_t^p(L_x^q)$, $(1 < p < \infty)$ - Banach spaces $L^p(\mathbb{R} : L^q(\mathbb{R}))$ for variables t and x respectively
- C, c - various constants whose exact values are immaterial
- $A \lesssim B$ - there exists a constant $C > 0$ such that $A < CB$
- $A \gtrsim B$ - there exists a constant $C > 0$ such that $A > CB$
- $A \sim B$ - $A \lesssim B$ and $A \gtrsim B$

- $\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}, \quad 1 \le p < \infty$, with usual modification for $p = \infty$
- $X^s := H^s(\mathbb{R}) \times H^s(\mathbb{R})$ - Cartesian product of Sobolev spaces
- $X := L^2(\mathbb{R}) \times L^2(\mathbb{R})$ - Cartesian product of L^2 spaces
- $\|\mathbf{f}\|_{X^s}^2 := \|f\|_{H^s}^2 + \|g\|_{H^s}^2$ for $\mathbf{f} = (f, g)$
- $\|\mathbf{f}\|_{L^p \times L^p} = \|f\|_{L^p} + \|g\|_{L^p}$
- $\|f\|_{L^p_x L^q_T} := \left(\int_{\mathbb{R}} \left(\int_0^T |f(x,t)|^q \, dt\right)^{p/q} \, dx\right)^{1/p}$ - mixed $L^p_x L^q_T$-norm with usual modification for $p = \infty$
- $\|\mathbf{f}\|_{L^p_x L^q_T} = \|f\|_{L^p_x L^q_T} + \|g\|_{L^p_x L^q_T}$
- $\operatorname{supp} \mathrm{f}$ - support of f
- $f * g$ - convolution product of f and g
- $a+ := a + \epsilon$ for $\epsilon > 0$

Chapter 1

Introduction

The subject nonlinear partial differential equations, in particular, nonlinear evolution equations of dispersive type is a very active field of research and study in recent times. Here we are going to address some recent developments in this field, viz., the well-posedness and the unique continuation property of the associated Cauchy problem, taking special considerations to some particular models that belong to this class. We start by defining what a dispersive equation means.

1.1 Dispersive Equations

Consider the following abstract form,

$$u_t = F(t, u). \tag{1.1.1}$$

of an evolution model, that may be used to describe many phenomena in physics (electron-plasma, ion field interaction, electromagnetic waves, gravitational waves, optical fibers etc.) or in fluid dynamics (water wave theory).

As the title suggests itself, we will concentrate our discussion on the dispersive models that belong to the class (1.1.1). In what follows we present a detailed account that leads to the definition of the dispersive models.

Consider the linear wave equations of the form

$$P(\partial_x, \partial_t)u = 0, \tag{1.1.2}$$

where P is a polynomial and ∂_x and ∂_t connote partial derivatives with respect to the space variable x and the time variable t respectively.

The plane wave solution to the equation (1.1.2) is of the form

$$u(x,t) = Ae^{i(kx-\omega t)}, \tag{1.1.3}$$

where A, k and ω represent the amplitude, the wave number and the frequency respectively.

One can write (1.1.3) as

$$u(x,t) = Ae^{ik(x-(\omega/k)t)}$$

to obtain the travelling wave solution to (1.1.2) with velocity ω/k.

Note that, u given in (1.1.3) will be a solution of (1.1.2) if and only if the following relation

$$P(ik, -i\omega) = 0, \tag{1.1.4}$$

is satisfied.

The relation between ω and k given by (1.1.4) is called the *Dispersion Relation*, from which one can get a formula determining the frequency $\omega = \omega(k)$ associated to the wave number k. Moreover, the phase velocity $c_p(k)$ and the group velocity c_g are defined by the follwing formulae.

$$c_p(k) = \frac{\omega}{k} \qquad \text{and} \qquad c_g = \frac{d\omega}{dk}. \tag{1.1.5}$$

The waves are called dispersive if the group velocity $c_g = \omega'(k)$ is not a constant quantity, i.e., if the following inequality

$$\omega''(k) \neq 0, \tag{1.1.6}$$

holds true.

It means that when time evolves the different waves disperse in the medium with the result that a single hump breaks into wave-trains (high-frequency waves travel much faster than low-frequency waves).

1.1.1 Examples

First we present some examples of the linear dispersive models.

1. The linear Schrödinger equation

$$i\partial_t u + \partial_x^2 u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{1.1.7}$$

   where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$. In this case, $\omega(k) = k^2,\ \omega''(k) = 2$.

2. The Airy equation (linear Korteweg-de Vries equation)

$$\partial_t u + \partial_x^3 u = 0, \quad x, t \in \mathbb{R}, \tag{1.1.8}$$

   where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this case, $\omega(k) = -k^3,\ \omega''(k) = -6k$.

3. The Boussinesq equation

$$\partial_t^2 v + \partial_x^4 v - \partial_x^2 v = 0, \quad x, t \in \mathbb{R}, \tag{1.1.9}$$

   where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this case, $\omega(k) = |k|\sqrt{1+k^2},\ \ \omega''(k) \neq 0$.

Now we record examples of some prominent nonlinear dispersive equations. The first one is the following.

1. The cubic nonlinear Schrödinger (NLS) equation

$$i\partial_t u + \Delta u = |u|^2 u, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1.1.10}$$

   where $u = u(x,t)$ is a complex valued function and Δ is a Laplacian; which arises naturally in quantum mechanics and nonlinear optics. Various physical phenomena are governed by equation (1.1.10), when $1 \leq n \leq 3$. For example, Langmuir waves in plasma, laser in optical fiber, water waves and many more are described by (1.1.10).

   The next important example is the following.

2. The Korteweg-de Vries (KdV) equation

$$\partial_t u + \partial_x^3 u = 6uu_x, \quad x, t \in \mathbb{R}, \tag{1.1.11}$$

where $u = u(x,t)$ is a real valued function. This equation arises in various contexts, for example, as a model to describe propagation of water waves with small amplitude in a swallow horizontal canal, interaction of internal waves in stratified fluids, propagation of gravitational waves, ion-acoustic waves in plasma etc. It has very rich mathematical structure and can also be solved by using inverse scattering technique.

1.1.2 Remarks

- The dispersive models that we described above are evolution equation, so natural problem to solve is the associated Cauchy problem.

- These are the Hamiltonian equations. Recall that a Hamiltonian ODE flow on the phase space $\mathbb{R}^{2n}$ has the form
$$\dot{u} = J\nabla H(u) \tag{1.1.12}$$
where $H : \mathbb{R}^{2n} \to \mathbb{R}$ is Hamiltonian, J is the matrix of the symplectic form.

  For the NLS equation these are given by
$$\langle Ju, v\rangle := \mathrm{Im}\int u\bar{v}\,dx, \quad H(u) := \int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4. \tag{1.1.13}$$

  For the KdV equation, we have
$$\langle Ju, v\rangle := \int u\Big(\frac{d}{dx}\Big)^{-1} v\,dx, \quad H(u) := \int \frac{1}{2}u_x^2 - \frac{1}{3}u^3. \tag{1.1.14}$$

  Conserved quantities: Hamiltonian is conserved by the flow, i.e., it is a time independent quantity. For the NLS and the KdV equations, L^2-norm
$$\int |u|^2\,dx \tag{1.1.15}$$
is also conserved. Let us note that, the conserved quantities play an important role to obtain solution for the long time period, it will be discussed later.

- Although non-linear, these models are only mild perturbation of the linear equation. In fact, they are semi-linear, i.e., they are of the form

$$Lu = F(u), \tag{1.1.16}$$

where L is linear evolution operator and $F(u)$ is purely nonlinear term which is of lower order than L.

- These equations are invariant under translation in both space and time. This suggests that the Fourier transform

$$\widehat{u}(\xi, t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-x\cdot\xi} u(x,t)\, dx \tag{1.1.17}$$

will be a useful tool to get solution. The Inverse Fourier transform is given by

$$u(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{x\cdot\xi} \widehat{u}(\xi, t)\, d\xi. \tag{1.1.18}$$

Example: Using the Fourier transform, it can be easily seen that the free Schrödinger equation

$$u_t + \Delta u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}$$

with the initial data

$$u(x,0) = e^{i\xi\cdot x} a(x),$$

where $a(x)$ is a bump function; evolves as

$$u(x,t) \sim e^{i\xi\cdot x} e^{-i|\xi|^2 t} a(x - 2\xi t).$$

So, a wave packet with frequency ξ travels at velocity 2ξ and retains its frequency. In the case of the linear KdV equation, a wave packet with frequency ξ travels at velocity 3ξ. In this sense, we say that the KdV equation possesses more dispersion than the Schrödinger equation.

1.2 Cauchy Problem

As mentioned earlier, we study the initial value problem (IVP) or the Cauchy problem

$$\begin{cases} Lu = F(u) \\ u(x,0) = u_0(x) \in X, \end{cases} \tag{1.2.19}$$

where L is an evolution operator of dispersive type and the initial data u_0 is given in a suitable function space X.

Before going further, let us define the concept of the well-posedness for the Cauchy problem that we are ging to use through out this text.

Definition 1 (Local well-posedness)**.** *The IVP* (1.2.19) *is said to be locally well-posed in a Banach space X, if for a given $u_0 \in X$, there exist a time $T > 0$ and a unique solution $u \in C([0,T];X)$ which depends continuously upon the initial data and satisfies the persistence property, it means, for given data $u_0 \in X$ the solution $u(t) \in X$ for all $t \in [-T,T]$ describes a continuous curve in X. If any one of the above conditions fails to hold we say that the IVP is ill-posed.*

Definition 2 (Global well-posedness)**.** *The IVP* (1.2.19) *is said to be globally well-posed in a Banach space X if the local in time solution can be extended to any time interval.*

In our case, the Banach space X will be the L^2-based Sobolev Space

$$H^s(\mathbb{R}) := \{f \in S'(\mathbb{R}) : (1+|\xi|^2)^{s/2}\hat{f} \in L^2(\mathbb{R})\}, \tag{1.2.20}$$

with norm defined by

$$\|f\|_{H^s} = \Big(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi\Big)^{1/2} \equiv \|\langle\xi\rangle^s \widehat{f}\|_{L^2_\xi}, \tag{1.2.21}$$

where S' is the space of the tempered distributions, $s \in \mathbb{R}$, $\hat{f}$ denotes the Fourier transform of f as defined in (1.1.17) and $\langle\xi\rangle = 1 + |\xi|$.

Just to recall, s is called the Sobolev index and measures the regularity of the function in H^s. The higher the value of the Sobolev index s, the more regular the function becomes.

We also define the homogeneous Sobolev space $\dot{H}^s$ with norm

$$\|f\|_{\dot{H}^s} = \Big(\int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \Big)^{1/2}. \tag{1.2.22}$$

In sequel, we record some more notations that will be used in this text. For $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ we define the mixed $L_x^p L_T^q$-norm by

$$\|f\|_{L_x^p L_T^q} = \left(\int_{\mathbb{R}} \left(\int_0^T |f(x,t)|^q \, dt \right)^{p/q} dx \right)^{1/p}, \tag{1.2.23}$$

with usual modifications when $p = \infty$. We replace T by t if $[0,T]$ is the whole real line $\mathbb{R}$.

We use c to denote various constants whose exact values are immaterial and may vary from one line to the next. We use $A \lesssim B$ to denote an estimate of the form $A \leq cB$ and $A \sim B$ if $A \leq cB$ and $B \leq cA$. Also, we use the notation $a+$ to denote $a+\epsilon$ for $\epsilon > 0$ and similar for $a-$.

Now, we introduce the concept of the scaling argument which helps a bit to get an idea about regularity of the initial data to have well-posedness results in the Sobolev spaces.

Scaling: If $u(x,t)$ is a solution then for any $\lambda > 0$,

$$u^\lambda(x,t) := \lambda^{-\alpha} u\Big(\frac{x}{\lambda}, \frac{t}{\lambda^\beta}\Big)$$

is also a solution.

For the KdV equation $(\alpha, \beta) = (2,3)$, and for the NLS equation $(\alpha, \beta) = (1,2)$. Note that,

$$\|u^\lambda\|_{\dot{H}^s} = \lambda^{\frac{n}{2} - \alpha - s} \|u\|_{\dot{H}^s}. \tag{1.2.24}$$

Critical regularity: The regularity s for which the norm $\|u\|_{\dot{H}^s}$ remains invariant under scaling is called the critical regularity. From (1.2.24) we see that the critical regularity is given by

$s_c := \frac{n}{2} - \alpha - s$. For the KdV equation $s_c = -\frac{3}{2}$ and for the NLS equation $s_c = \frac{n}{2} - 1$.

One may expect that the IVP associated to the models considered here could be well-posed for the data in the Sobolev spaces H^s for $s \geq s_c$, where s_c is the critical regularity. Therefore, it is clear that the regularity s at which one may expect local well-posedness for the KdV equation is $s \geq -3/2$ and that for the NLS equation is $s \geq n/2 - 1$.

The objective of this work is two fold. First we focus on the local and global well-posedness issues of the Cauchy problem associated to some nonlinear dispersive systems. Our second objective is to address the unique continuation property of some bi-dimensional versions of the KdV equation. In recent years, this field has become quite active, with a substantial amount of work being developed by notable mathematicians such as J. Bourgain, C. Kenig, G. Ponce, L. Vega, S. Klainerman and T. Tao (see [5]-[13], [25], [24], [46]-[60] and references therein).

There is a large number of interesting nonlinear evolution equations of dispersive type describing different physical phenomena. Although a limited amount of generalization is possible, each individual PDE typically has its own structure (geometric invariance, conserved quantities, etc..), requiring separate treatment, specially when dealing with the delicate issue of global existence of large initial data of low regularity. With a few notable exceptions (KdV, mKdV, 1-d cubic NLS), the majority of the dispersive equations are not completely integrable, and almost certainly not reducible via algebraic transformations to a linear evolution equation; thus there is essentially no hope of finding exact solutions to these equations from general initial data via some basic algebraic formula, although there are certainly many important special exact solutions, like solitons, which play major roles in the subject and provide important examples and intuition. In the absence of exact formulae for general solutions, the analytical theory instead revolves around qualitative and quantitative properties of the solutions. Qualitative properties include the fundamental question of well-posedness. Quantitative properties typically involve estimating various spatial or spacetime norms of the solution in terms of various norms of the initial data.

One of the main objectives of the analytical theory of these equations is to rigorously classify, based on the equation and on the class of the initial data, whether the global evolution of the equation exhibits linear or nonlinear behavior. Although, in many applications, it is the smooth (high regularity) solutions which are of importance, it is often worthwhile to fully develop the low regularity theory, as the estimates obtained as a consequence of that theory are often extremely useful in controlling the global and asymptotic behavior of smooth solutions, and in particular in obtaining precise criteria as to whether blowup or other bad behavior will occur from smooth initial data. As noted earlier, our interest is in studying the local and global well-posedness issues for the low regularity data. The principal tools to accomplish this are the quantitative estimates obtained from harmonic analysis techniques, developed recently by Bourgain [17], Kenig-Ponce-Vega ([52] - [47]) and T. Tao [86].

There are several techniques to prove local well-posedness if one considers enough regularity in the initial data. Conserved quantities satisfied by the flow of the model under consideration provide an *a priori* estimate that helps to extend local solution globally in time in certain class of Sobolev spaces. However, if the initial data has no enough regularity, due to lack of conserved quantities it is quite demanding to prove global well-posedness for such data.

Fourier transform restriction norm method [17, 51] has been proved to be a powerful tool to obtain local well-posedness and the I-method (or the method of almost conserved quantities) [25, 24, 86], to obtain global well-posedness for data with low Sobolev regularity. In this work, our focus will be on the I-method to prove global well-posedness. The I-method is a method for constructing global solutions to nonlinear dispersive equations in situations when the relevant conserved quantity (such as the energy $E(u)$) is subcritical but infinite. One applies a mollifying operator I to the solution (dependent on a large frequency truncation parameter N) to make the quantity $E(Iu)$ finite. The point is that this quantity is no longer conserved, but one hopes to show an almost conservation law for this quantity which makes it stable over long periods of time. Letting N go to infinity one obtains global well-posedness in certain range of Sobolev regularity of the initial data.

In what follows we explain in brief, some methods that ares used to address the well-posedness issues for the Cauchy problem associated to the dispersive equations.

1.2.1 Solution Method

In this section we discuss in brief about the methods of solutions and recent developments in this direction with special emphasis to the Cauchy problem associated to the KdV model.

Viscosity method

This is the most classical method to obtain solution to the Cauchy problem associated to the KdV and NLS equations and is also known as parabolic regularization. For instance, using this method to solve the IVP associated to the KdV equation, one considers for $\epsilon > 0$,

$$\begin{cases} u_t + u_{xxx} + uu_x - \epsilon u_{xx} = 0 \\ u(x,0) = u_0(x). \end{cases} \tag{1.2.25}$$

Now the IVP (1.2.25) turns out to be a parabolic equation and can be solved by the theory developed for this class of equations. finally, taking limit $\epsilon \to 0$ in an appropriate space gives solution to the original problem.

Iteration (Picard) method

In this method, one treats the nonlinear term $F(u)$ of the Cauchy problem as a perturbation. So, using this method, one solves the linear problem

$$Lu^{(0)} = 0, \quad u^{(0)}(0) = u_0,$$

and constructs successive approximations $u^{(1)}, u^{(2)}, \cdots$ by solving inhomogeneous linear problems

$$Lu^{(n+1)} = F(u^{(n)}), \quad u^{(n+1)}(0) = u_0.$$

The sequence $u^{(n)}$ converges to u in a suitable space which solves the original problem.

This method is applicable to several dispersive models. Let us explain this method with the help of a simple example in the ordinary differential equation (ODE) case.

Example: Consider the following scalar ODE

$$\dot{u} = u^2, \quad u(0) = 2. \tag{1.2.26}$$

Using the Fundamental Theorem of Calculus we can write (1.2.26) as

$$u(t) = 2 + \int_0^t u^2(t')\,dt'. \tag{1.2.27}$$

Let us define an application given by

$$\Phi(u(t)) = 2 + \int_0^t u^2(t')\,dt'. \tag{1.2.28}$$

Note that, the fixed point of Φ will be a solution to the IVP (1.2.26). Therefore, the idea is to show that the application Φ is a contraction map on a ball of radius, say a, in $L^\infty(0,T)$. We can do it, if T is sufficiently small. For,

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{L^\infty([0,T])} &\leq \int_0^T |u^2(t') - v^2(t')|dt' \\ &\leq T\|u+v\|_\infty \|u-v\|_\infty \\ &\leq 2aT\|u-v\|_\infty.\end{aligned} \tag{1.2.29}$$

Now, if we choose $T > 0$ such that $2aT < 1$, then we can get the desired contraction map Φ from (1.2.29). This technique can also be applied to a wide range of the dispersive PDEs too.

To prove the local well-posedness for the IVP (1.2.19), we use Duhamel's formula to write it in an equivalent integral form

$$u(t) = U(t)u_0 + \int_0^t U(t-t')F(u(t'))\,dt', \tag{1.2.30}$$

where $U(t)$ is the semi-group operator associated to the solution of the linear problem. Now, we define an application Ψ by

$$\Psi(u)(t) = U(t)u_0 + \int_0^t U(t-t')F(u(t'))\,dt'. \tag{1.2.31}$$

As in the ODE case, here too, the objective is to perform an iteration in a ball $\mathcal{B}_a$ in $L^\infty([0,T];H^s(\mathbb{R}))$. For this, the following three steps are essential.

- Prove that there is a Banach space X^s and a number $s_0 \in \mathbb{R}$, $s \geq s_0$ such that $X^s \subset C([0,T]; H^s)$ and for some $\alpha > 0$,

$$\|U(t)u_0\|_{X^s} \leq c_0\|u_0\|_{H^s} \tag{1.2.32}$$

$$\Big\| \int_0^t U(t-t')F(u(t'))dt' \Big\|_{X^s} \leq cT^\alpha G(\|u\|_{X^{s_0}})\|u\|_{X^s}. \tag{1.2.33}$$

$$\Big\| \int_0^t U(t-t')[F(u)-F(v)](t')dt' \Big\|_{X^s} \leq cT^\alpha \max\{G(\|u\|_{X^{s_0}}), \tilde{G}(\|u\|_{X^{s_0}})\}\|u-v\|_{X^s}, \tag{1.2.34}$$

where $G, \tilde{G} : \mathbb{R} \to \mathbb{R}^+$ are increasing and bounded on bounded sets.

- Take $a = 2c_0\|u_0\|_{H^s}$, and $T = c\|u_0\|^{-\frac{1}{\alpha}}$

- Prove that $\Psi : \mathcal{B}_a \subset X^s \to \mathcal{B}_a$ is a contraction map.

Remark 1.1. *In the above argument, the existence of the positive exponent α in* (1.2.33) *is quite important. In some situations, α may be zero and our argument as it is does not work. But nothing has gone bad, we can overcome this situation with scaling argument (if it applies) or by using a smooth cut-off function in time. For this, we define a cut-off function $\psi_1 \in C^\infty(\mathbb{R};\mathbb{R}^+)$ that is even such that $0 \leq \psi_1 \leq 1$ and*

$$\psi_1(t) = \begin{cases} 1, & |t| \leq 1 \\ 0, & |t| \geq 2. \end{cases} \tag{1.2.35}$$

We also define $\psi_T(t) = \psi_1(t/T)$, for $0 \leq T \leq 1$.

As we interested in getting the local well-posedness, existence of any positive time will be enough for our purpose. So we can replace the application (1.2.31) *by the following one*

$$\Psi(u)(t) = \psi_1(t)U(t)u_0 + \psi_T(t)\int_0^t U(t-t')F(u(t'))\,dt'. \tag{1.2.36}$$

Now the cut-off function $\psi_T(t)$ in (1.2.36) *helps to obtain the desired time factor in* (1.2.33).

In what follows, we explain in brief, how this process is carried out to address the IVP associated to the KdV equation

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2.37}$$

As discussed above, finding an appropriate space X^s to perform a contraction mapping argument is very crucial. In sequel, we state some estimates satisfied by the solution to the KdV equation that lead one to define such a space. Although the proofs of some estimates are simple, the others are quite involving. Here we will just state them indicating the references where the proofs can be found.

We start by stating the local smoothing effect due to Kato [46]:

$$\int_{|x|\le R} \int_{|t|\le R} |\partial_x u(x,t)|^2 \, dxdt \le C(R, \|u_0\|_{L^2}).$$

The next is the smoothing effect due to Kenig-Ponce-Vega [54, 55]:

$$\|\partial_x U(t) u_0\|_{L^\infty_x L^2_t} \le c \|u_0\|_{L^2}, \tag{1.2.38}$$

and its inhomogeneous version, also proved by Kenig-Ponce-Vega [52]:

$$\|\partial_x^2 \int_o^t U(t-t') f(\cdot, t') \, dt' \|_{L^\infty_x L^2_t} \le c \|f\|_{L^1_x L^2_t}. \tag{1.2.39}$$

Also, for the KdV equation, the following Strichartz estimate holds whose proof can be found in [56]:

For $(\theta, \alpha) \in [0,1] \times [0, 1/2]$, $(q,p) = \big(\frac{6}{\theta(\alpha+1)}, \frac{2}{1-\theta}\big)$, one has

$$\|D^{\theta\alpha/2} U(t) u_0\|_{L^q_t L^p_x} \le c \|u_0\|_{L^2}. \tag{1.2.40}$$

The following is the inhomogeneous version of (1.2.40)

$$\|D^{\theta\alpha} \int_0^t U(t-t') f(\cdot, t') \|_{L^q_t L^p_x} \le c \|f\|_{L^{q'}_t L^{p'}_x}. \tag{1.2.41}$$

Now, we record the maximal function estimate obtained by Kenig-Ponce-Vega [54]:

If, $0 \leq T \leq 1$, $s > 3/4$ and $\rho > 1/4$, then

$$\|U(t)u_0\|_{L^2_x L^\infty_T} \leq (1+T)^\rho \|u_0\|_{H^s}, \tag{1.2.42}$$

and its sharp version proved in [52]:

$$\|U(t)u_0\|_{L^4_x L^\infty_T} \leq c\|D^{1/4}u_0\|_{L^2}. \tag{1.2.43}$$

Finally, we record the Leibniz's rule for fractional derivatives, whose proof can be found in Kenig-Ponce-Vega [52]:

Let $\alpha \in (0,1)$, $\alpha_1 + \alpha_2 = \alpha$ then

$$\|D^\alpha_x(fg) - fD^\alpha_x g - gD^\alpha_x f\|_{L^p_x L^q_T} \leq c\|D^{\alpha_1}_x f\|_{L^{p_1}_x L^{q_1}_T} \|D^{\alpha_2}_x g\|_{L^{p_2}_x L^{q_2}_T} \tag{1.2.44}$$

For $\alpha_1 = 0$, $q_1 = \infty$ is allowed in (1.2.44).

With the above estimates in hand, for the KdV equation, the space X^s is defined via the norm

$$\|u\|_{X^s} := \|u\|_{L^\infty_T H^s} + \|\partial_x u\|_{L^4_T L^\infty_x} + \|D^s \partial_x u\|_{L^\infty_x L^2_T} + \|u\|_{L^2_x L^\infty_T}. \tag{1.2.45}$$

Using this machinery, Kenig-Ponce-Vega [52] proved that the Cauchy problem associated to the KdV equation is locally well-posed for given data in $H^s(\mathbb{R})$, $s > 3/4$. Also, using the conserved quantities (1.1.14) and (1.1.15), it is easy to get an *a priori* estimate in H^1, which allows to get global well-posedness in $H^s(\mathbb{R})$, $s \geq 1$.

To obtain well-posedness result for the data with regularity less than $\frac{3}{4}$, Bourgain in [17] introduced a new method that we describe below.

Fourier transform restriction norm method

For $s, b \in \mathbb{R}$, Bourgain [17], introduced the space $X_{s,b}$ with the norm

$$\|f\|_{X_{s,b}}^2 = \int_{\mathbb{R}^2} (1+|\xi|)^{2s}(1+|\tau-\xi^3|)^{2b}|\widetilde{f}(\xi,\tau)|^2\, d\xi d\tau, \tag{1.2.46}$$

where $\widetilde{f}$ given by

$$\widetilde{f}(\xi,\tau) = (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-i(x\xi+t\tau)}f(x,t)\, dx\, dt,$$

denotes the Fourier transform of f in both x and t variables.

The $X_{s,b}$ norm can also be written as

$$\|f\|_{X_{s,b}} = \|(1+D_t)^b U(t) f\|_{L_t^2(H_x^s)} = \|\langle\tau-\xi^3\rangle^b\langle\xi\rangle^s\widetilde{f}\|_{L^2_{\xi,\tau}}, \tag{1.2.47}$$

where $U(t)$ is the unitary group associated with the linear problem. In above definitions we have used the unitary group $U(t) = e^{-t\partial_x^3}$ associated to the linear KdV equation and one can equally define for other dispersive models too.

With this norm, the space $X_{s,b}$ becomes a Hilbert space. If $b > 1/2$, the Sobolev lemma implies that,

$$X_{s,b} \subset C(\mathbb{R}; H_x^s(\mathbb{R})).$$

For any interval I, we define the localized spaces $X_{s,b}^I := X_{s,b}(\mathbb{R}\times I)$ with norm

$$\|f\|_{X_{s,b}(\mathbb{R}\times I)} = \inf\left\{\|g\|_{X_{s,b}} : g|_{\mathbb{R}\times I} = f\right\}. \tag{1.2.48}$$

Sometimes we use the definition $X_{s,b}^\delta := \|f\|_{X_{s,b}(\mathbb{R}\times[0,\delta])}$.

Here are some inclusion estimates related to $X_{s,b}$ spaces: for $s_1 \leq s_2$ and $b_1 \leq b_2$ we have $X_{s_2,b_2} \subseteq X_{s_1,b_1}$. Moreover, we have the following relations.

$$\|u\|_{L_x^4 L_t^\infty} \lesssim \|u\|_{X_{s,b}^I}, \qquad s \geq \frac{1}{4} \tag{1.2.49}$$

$$\|u\|_{L_x^8 L_t^8} \lesssim \|u\|_{X_{0,b}^I}, \tag{1.2.50}$$

$$\|u\|_{L_x^6 L_t^6} \lesssim \|u\|_{X_{0,b}^I}, \tag{1.2.51}$$

In what follows we list some basic estimates in the $X_{s,b}$ spaces that are crucial in the proof of the local well-posedness result.

Lemma 1.1. *For any* $s, b \in \mathbb{R}$,

$$\|\psi_1 U(t)\phi\|_{X_{s,b}} \leq C\|\phi\|_{H^s}. \tag{1.2.52}$$

Let $-\frac{1}{2} < b' \leq 0 \leq b < b'+1$ *and* $0 \leq \delta \leq 1$*, then*

$$\|\psi_T \int_0^t U(t-t')f(u(t'))dt'\|_{X_{s,b}} \leq C\delta^{1-b-b'}\|f(u)\|_{X_{s,b'}}. \tag{1.2.53}$$

Proof. For an easy proof of these estimates we refer to [31]. □

The following is the key bilinear estimate for the KdV equation due to Kenig-Ponce-Vega [51],

$$\|uu_x\|_{X_{s,b'}} \leq c\|u\|_{X_{s,b}}^2 \qquad b > \frac{1}{2},\ b' > -\frac{1}{2},\ s > -\frac{3}{4}. \tag{1.2.54}$$

Also we have the following trilinear estimate due to Tao [86],

$$\|(u^3)_x\|_{X_{s,b'}} \leq C\|u\|_{X_{s,b}}^3, \qquad b > \frac{1}{2},\ b' > -\frac{1}{2},\ s \geq \frac{1}{4}. \tag{1.2.55}$$

Bourgain [17] proved the estimate (1.2.54) for $s \geq 0$, thereby obtaining local well-posedness for the given data in $H^s(\mathbb{R})$, $s \geq 0$. Also, the L^2-conserved quantity (1.1.15) implies global well-posedness in the same spaces. Later, Kenig-Ponce-Vega [51] improved the result of Bourgain by showing that the estimate (1.2.54) holds for $s > -\frac{3}{4}$ which leads to the local well-posedness of the IVP associated to the KdV equation for data in $H^s(\mathbb{R})$, $s > -\frac{3}{4}$. Also, they proved that the estimate (1.2.54) fails for $s < -3/4$. Note that, this method also applies to the NLS equation and in the periodic case too. As there are no conserved quantities to get an *a priori* estimate in the Sobolev spaces of negative index, the global well-posedness in such spaces needs to be addressed with different techniques. One such method is the method of almost conserved quantities or *I-method* introduced by Colliander et al [25, 24] that we described a bit in earlier discussions. This is one of the method we will employ in this work to address the global well-posedness of the models we consider.

1.3 Nonlinear Dispersive Systems

In this section we introduce the following system of nonlinear dispersive equations

$$\begin{cases} W_t + AW_{xxx} + B(W)W_x + CW_x = 0, \qquad x,\, t \in \mathbb{R} \\ W(x,0) = W_0(x), \end{cases} \tag{1.3.56}$$

where $W = (u, v)^t$ with $u = u(x,t)$ and $v = v(x,t)$, real valued functions. The typical examples of the models we want to consider are the coupled system of the Korteweg-de Vries (KdV) equations. The above model arises in various physical contexts to describe several nonlinear phenomena.

Our main purpose here is to address the well-posedness issues to the initial value problem (IVP) (1.3.56) in some special cases of matrices A, $B(W)$ and C.

A large amount of work has been devoted to study (1.3.56). For example, when

$$A = \begin{pmatrix} 1 & a_3 \\ \frac{b_2 a_3}{b_1} & \frac{1}{b_1} \end{pmatrix}, \quad B(W) = \begin{pmatrix} u + a_2 v & a_2 u + a_1 v \\ \frac{b_2 a_2}{b_1} u + \frac{b_2 a_1}{b_1} v & \frac{b_2 a_1}{b_1} u + \frac{1}{b_1} v \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & \frac{r}{b_1} \end{pmatrix},$$

with $a_1, a_2, a_3, r \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}^+$, the system (1.3.56) is a well known model introduced by Gear and Grimshaw [30] to describe the strong interaction of two dimensional, long, internal gravity waves propagating on a neighboring pycnoclines in a stratified fluid.

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 v v_x + a_2 (uv)_x = 0, \qquad x,\, t \in \mathbb{R}, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_2 u u_x + b_2 a_1 (uv)_x + r v_x = 0, \\ u(x,0) = \phi(x), \\ v(x,0) = \psi(x). \end{cases} \tag{1.3.57}$$

The system (1.3.57) has the structure of the KdV equation

$$u_t + u_{xxx} + uu_x = 0, \qquad x,\, t \in \mathbb{R}, \tag{1.3.58}$$

coupled through dispersive as well as nonlinear effects. Several properties of the system (1.3.57) including existence theory for the associated IVP and the existence and stability of solitary wave solution can be found in the literature. For an extensive description of this model we refer to the work of Bona, Ponce, Saut and Tom [9]. They used Kato's theory for abstract evolution equations to obtain well-posedness results in classical Sobolev spaces. Further they utilized the theory developed by Kenig, Ponce and Vega [54] to get the local result in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

Gear and Grimshaw [30] showed that the following quantities

$$\int u\,dx, \qquad \int v\,dx \qquad \text{and} \qquad \int (b_2u^2 + b_1v^2)\,dx$$

are conserved by the flow of (1.3.57). Bona et. al. [9] derived a new conserved quantity

$$\int \left\{b_2(u_x^2 + 2a_3u_xv_x - \frac{1}{3}u^3 - a_2u^2v - a_1uv^2) + v_x^2 - \frac{1}{3}v^3\right\} dx.$$

Using these four conserved quantities they were able to get an *a priori* estimate in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ by imposing the condition $1 - b_2a_3^2 > 0$ on the coefficients. This *a priori* estimate permits one to extend the local solution to a global one. This result is obtained by neglecting the dimensionless parameter r. Later, Ash, Cohen and Wang [5] studied this problem in the $X_{s,b}$ spaces introduced by Bourgain to deal with the nonlinear dispersive equations. Using bilinear estimates established by Kenig, Ponce and Vega [51] they proved local well-posedness for given data in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Also, they utilized the L^2-conserved quantity to extend the local solution to the global one in that space. Recently, Saut and Tzvetkov [74] considered the IVP (1.3.57) without neglecting the constant r and proved global well-posedness in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Using the concentration compactness technique, Bona and Chen [8] proved the existence of solitary waves for the system (1.3.57) as global minimizers to the constrained variational problem. Later, Albert and Linares [3] proved that the solitary waves are stable in a weak sense by considering $a_3^2b_2 < 1$. Recently, Menzala, Vasconcellos and Zuazua [62] showed that the solutions of the KdV equation in a bounded interval under the effect of a localized damping decay exponentially in time. The method of proof is a combination of multiplier techniques, compactness arguments and the unique continuation property of the KdV equation. A similar result for the IVP (1.3.57) was obtained by Bisognin, Bisognin and Menzala [7] whenever the conditions $b_1 = b_2$ and $0 < a_3 < 1$ hold.

Since (1.3.57) is a coupled system of KdV equations, it is natural to ask whether it shares similar results like KdV equations. In other words, whether we can lower the Sobolev index in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ as in the case of the KdV equation to get well-posedness results? Using the scaling argument we can have an insight to this question. Observe that if (u, v) solves (1.3.57) (note that we are neglecting the parameter r, otherwise the scaling doesn't work) with initial data (ϕ, ψ) then for $\lambda > 0$ so does (u^λ, v^λ) with initial data $(\phi^\lambda, \psi^\lambda)$; where $u^\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^3 t)$, $v^\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^3 t)$, $\phi^\lambda(x) = \lambda^2 \phi(\lambda x)$ and $\psi^\lambda(x) = \lambda^2 \psi(\lambda x)$. Note that,

$$\|(\phi^\lambda, \psi^\lambda)\|_{\dot{H}^s \times \dot{H}^s} = \lambda^{2s+3} \|(\phi, \psi)\|_{\dot{H}^s \times \dot{H}^s}, \tag{1.3.59}$$

where $\dot{H}^s(\mathbb{R})$ denotes the homogeneous Sobolev space of order s. Thus, we see from (1.3.59) that the well-posedness result for the IVP (1.3.57) could be achieved in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq -3/2$.

Bourgain [14] showed that the well-posedness result obtained by Kenig, Ponce and Vega [51] for the KdV equation in $H^s(\mathbb{R})$, $s > -3/4$, is essentially optimal if one strengthens the usual notion of well-posedness by requiring the flow-map

$$\phi \mapsto u_\phi(t), \qquad |t| < T$$

should act smoothly (for eg. C^3) on the space under consideration (instead of just continuous). This notion of well-posedness seems to be natural because, if one uses the contraction mapping principle to solve the integral equation associated with the Cauchy problem, the flow-map acts smoothly from H^s to itself. In fact, for $s > -3/4$ the flow-map is real analytic (see for eg., [51] [52] [17]). Takaoka [82] used this technique to show that the nonlinear Schrödinger equation with derivative in a nonlinear term is ill-posed in $H^s(\mathbb{R})$, $s < 1/2$. Further, Tzvetkov [90] showed that the KdV equation is locally ill-posed in $H^s(\mathbb{R})$ for $s < -3/4$ if one requires only C^2 regularity of the flow-map in the notion of well-posedness. Following the same scheme, we prove that the IVP (1.3.57) is ill-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s < -3/4$. This result is in agreement with the KdV results. So, one expects that the IVP (1.3.57) be locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/4$. In fact, using the bilinear estimate (1.2.54) established by Kenig, Ponce and Vega [51], we prove that the IVP (1.3.57) is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, whenever $s > -3/4$.

On the other hand, we should mention that, Nakanishi, Takaoka and Tsutsumi [68] constructed a counterexample to prove that the bilinear estimate established by Kenig, Ponce and Vega [51] fails when $s = -3/4$. Therefore the critical index $s = -3/4$ cannot be achieved using this method. However, Christ, Colliander and Tao [23] showed recently the existence of the solutions to the IVP associated to the KdV equation in $H^{-3/4}(\mathbb{R})$ using a generalized Miura transform to transfer the existing local theory for the modified KdV equation in $H^{1/4}(\mathbb{R})$.

Note that, in the Sobolev spaces of negative index, conservation laws are not available to extend the local solution to a global one. To overcome this difficulty, i.e. lack of conservation laws, quite recently, Colliander, Keel, Staffilani, Takaoka and Tao [25] introduced a variant of Bourgain's method [13] called *I-method* to obtain a global solution to KdV equation in Sobolev spaces of negative index. For this, they introduced a notion of an *almost conserved quantity* by utilizing an appropriate Fourier multiplier operator I. To obtain such almost conserved quantity they exploited some internal cancellation which the KdV equation satisfies. The cancellation plays a main role in this process. In our case, it is not possible to get such cancellation unless the coefficients $a_3 = 0$ and $b_1 = b_2$ (see Lemma 2.3 below). It seems that we cannot get more cancellation in the general case because the IVP under consideration is not completely integrable. Using this method, under above conditions on the coefficients, we prove that the IVP (1.3.57) is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/10$.

For $p \in \mathbb{Z}^+$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B(W) = \begin{pmatrix} \frac{1}{p}u^{p-1}v^{p+1} & \frac{1}{p+1}u^p v^p \\ \frac{1}{p+1}u^p v^p & \frac{1}{p}u^{p+1}v^{p-1} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the model (1.3.56) turns out to be a coupled system of generalized KdV equations

$$\begin{cases} u_t + u_{xxx} + (u^p v^{p+1})_x = 0 \\ v_t + v_{xxx} + (u^{p+1} v^p)_x = 0, \qquad x,\, t \in \mathbb{R} \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x), \end{cases} \tag{1.3.60}$$

and arises in various physical situations. This system has the following conserved quantities

$$I_1(u,v) = \int_{\mathbb{R}} (u^2 + v^2)\, dx \tag{1.3.61}$$

and

$$I_2(u,v) = \int_{\mathbb{R}} \left\{ u_x^2 + v_x^2 - \frac{2}{p+1} u^{p+1} v^{p+1} \right\} dx, \tag{1.3.62}$$

and admits sech solitary wave solutions. This model has been extensively studied in recent years. Alarcon, Angulo and Montenegro [2] considered the IVP (1.3.60) and proved local and global well-posedness results for given data $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$. To get global results they used the above conserved quantities satisfied by the flow of (1.3.60) along with some size restriction on the given data depending on the values of p. They also studied the existence and nonlinear stability of the solitary wave solution to this model from the point of view of the abstract theory of Grillakis, Shatah and Strauss [32]. In [2] the solitary wave solution to the system (1.3.60) were shown to be orbitally stable for $p < 2$ and unstable for $p > 2$. To obtain the instability result they followed a method established by Bona, Souganidis and Strauss [11] in the KdV context.

Some particular cases of the IVP (1.3.60) have also been a matter of interest in recent literature. When $p = 1$, this model reduces to a system of modified KdV (mKdV) equations

$$\begin{cases} u_t + u_{xxx} + (uv^2)_x = 0 \\ v_t + v_{xxx} + (u^2v)_x = 0, \qquad x,\, t \in \mathbb{R} \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x), \end{cases} \tag{1.3.63}$$

where $u = u(x,t)$ and $v = v(x,t)$ are real valued functions and is coupled through the nonlinear terms. Montenegro [67] used the theory developed by Kenig, Ponce and Vega [52] in the mKdV context to prove that the IVP associated to this particular model for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ has local solution when $s \geq 1/4$ and global solution when $s \geq 1$. So there is a gap in the Sobolev indices between the local and global existence results. In this work we will present the result proved in [70] that fills this gap to some extent. For this purpose, the method used in [70] is the refined use of the high-low frequency method introduced my Bourgain [13] and further modified by Fonseca et al [27, 28]. In what follows we describe this method.

Bourgain [13] introduced a new technique to get the global solution below energy spaces. Let us explain in brief about how to implement the Bourgain's technique in the case of a general dispersive equation,

$$\begin{cases} w_t + P(D)w + f(w) = 0 \\ w(x,0) = \phi(x). \end{cases} \tag{1.3.64}$$

Suppose that the IVP (1.3.64) has local solution in H^{s_0} for some $s_0 \in (0,1)$ and its flow satisfies the H^1 conservation law. To extend the local solution to the global one below the energy space H^1, we proceed as follows.

We decompose the initial data $\phi \in H^s,\ s<1$, to $\phi = \phi_1 + \psi_1$, where ϕ_1 and ψ_1 are given by,

$$\hat{\phi}_1(\xi) = \chi_{\{|\xi|\le N\}}\hat{\phi}(\xi), \qquad \hat{\psi}_1(\xi) = \chi_{\{|\xi|>N\}}\hat{\phi}(\xi).$$

In other words, we decompose ϕ into low and high frequency parts so that the low frequency part ϕ_1 is regular with H^1-norm large and the high frequency part ψ_1, although does not improve regularity, has H^s-norm small. In fact,

$$\|\phi_1\|_{H^1} \lesssim N^{1-s} \quad \text{and} \quad \|\psi_1\|_{\mathrm{H}^\rho} \lesssim \mathrm{N}^{\rho-\mathrm{s}}, \quad 0 \le \rho \le \mathrm{s} < 1.$$

We evolve the low frequency part ϕ_1 according to the original IVP (1.3.64) so that we have the existence result say in $[0,\delta]$. Let $\phi_1 \mapsto u_1(t)$ be the evolution of the low frequency part. Now we evolve the high frequency part ψ_1 according to the difference equation

$$\begin{cases} v_{1t} + P(D)v_1 + f(u_1+v_1) - f(u_1) = 0, & x,\, t \in \mathbb{R} \\ v_1(x,0) = \psi_1(x), \end{cases} \tag{1.3.65}$$

with variable coefficients depending on the solution u_1. For simplicity, let us denote the evolution of the high frequency part $\psi_1 \mapsto v_1(t)$ by $v_1(t) = U(t)\psi_1 + z_1(t)$, where $U(t)$ is the unitary group associated to the linear problem. The main feature of this technique is that the existence interval $[0,\delta]$ is the same for both u_1 and v_1 and $w(t) = u_1(t) + v_1(t)$ solves the IVP (1.3.64). Note that the inhomogeneous part $z_1(t)$ of the evolution $v_1(t)$ of the high frequency part ψ_1 is smoother than the data itself. In fact, for some $\alpha = \alpha(s) > 0$

$$\|z_1(t)\|_{H^1} \lesssim N^{-\alpha}. \tag{1.3.66}$$

Thus at time $t=\delta$ we add $z_1(\delta)$ to $u_1(\delta)$ and repeat the above argument with new data

$$\phi_2 := u_1(\delta) + z_1(\delta) \qquad \text{and} \qquad \psi_2 := U(\delta)\psi_1$$

to obtain the solution in $[\delta, 2\delta]$. Then we iterate this process to cover the time interval $[0, T]$ for arbitrary $T > 0$. In each step of iteration it is necessary to control the involved norms taking care of the contribution arising from (1.3.66) (also called as error term). In fact, we can proceed with this iteration process as long as the total error is at most comparable with the size of $\|\phi_1\|_{H^1}$ and at this point we obtain restriction on the Sobolev index s.

Soon after Bourgain [13] introduced this technique to get the global solution to the two-dimensional Schrödinger equation below energy space, several authors have applied it to obtain the global solution to various nonlinear dispersive models. Fonseca, Linares and Ponce [27] simplified this technique to get the global solution to the mKdV equation in $H^s(\mathbb{R})$, $s > 3/5$. It is also applied to get the global solutions to the semi-linear wave equations (see Kenig, Ponce and Vega [50]) and critical generalized KdV equations (see Fonseca, Linares and Ponce [28]). Also, Takaoka used this technique to get the global solutions to KP-II equation in [83] and to the Schrödinger equation with derivative in [82]. Further, Pecher [73] followed the same technique to prove the global well-posedness for the 1D Zakharov system below the energy space. Recently, using the argument in [27], Carvajal and Linares [20] proved that the IVP associated to the higher order nonlinear Schrödinger equation is globally well-posed in $H^s(\mathbb{R})$, $s > 5/9$.

Here we further refine this technique by exploiting the uniform bound of the solution (see (3.2.1) below) obtained by using iteration in the energy space. With proper choice of the Sobolev indices we develop an iteration process below the energy space and prove that the IVP (1.3.63) is globally well-posed for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 4/9$.

When $p = 2$, the IVP (1.3.60) turns out to be a coupled system of critical KdV (cKdV) equations, i.e.,

$$\begin{cases} u_t + \partial_x^3 u + \partial_x(u^2 v^3) = 0, \\ v_t + \partial_x^3 v + \partial_x(u^3 v^2) = 0, \qquad x,\, t \in \mathbb{R},\, p \in \mathbb{Z}^+ \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x). \end{cases} \tag{1.3.67}$$

We say the system (1.3.67) (i.e., (1.3.60) with $p = 2$) is critical because we have global solutions in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for all data when $p < 2$ and global solutions only for small data (i.e., data small in $H^1 \times H^1$-norm) when $p > 2$. Also, the solitary wave solutions are orbitally stable for $p < 2$ and unstable for $p > 2$. This feature for $p = 2$ resembles that of the critical generalized KdV equation

$$u_t + u_{xxx} + (u^k)_x = 0,$$

with $k = 5$. So, naming the system (1.3.67) critical seems well justified.

In the case of the critical KdV equation, the size restriction on the initial data needed to obtain global solutions in $H^1(\mathbb{R})$ is $\|u(0)\|_{L^2} < \|S\|_{L^2}$, where S is the solitary wave solution to the critical KdV equation. Merle in [63] proved that there exists $u_0 \in H^1$, with $\|u_0\|_{L^2} > \|S\|_{L^2}$, such that the corresponding solution to the critical KdV equation blows-up in finite time. We do not know whether we can have a result of blow-up solution in the case of system (1.3.67) with initial data $u_0 \neq v_0$.

Recently, exploiting the criticality of the system (1.3.67), Hakkaev and Kirchev in [33] studied the stability of the solitary wave solution. The authors in [33] used the ideas and techniques introduced by Angulo, Bona, Linares and Scialom in [4] to obtain analogous results to that for the critical KdV equation.

In this work we are interested in addressing some questions related to the well-posedness of the IVP (1.3.67) for given data in low regularity Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. We will improve the well-posedness results obtained by Alarcon, Angulo and Montenegro in [2].

1.4 Bi-dimensional Versions of the KdV Equation

The next models we want to address are the two dimensional generalizations of the KdV equation (1.3.58). The KdV model was obtained in [59] to describe the propagation of one dimensional surface gravity waves with small amplitude in a shallow channel of water. This is a widely studied model and arises in various physical contexts. There are two dimensional generalizations of the KdV model that arise to govern the motion where transversal effects are also taken into consideration. The most known two dimensional generalizations of the KdV equation are the Kadomtsev-Petviashvili (KP) equation

$$(u_t + u_{xxx} + uu_x)_x + \alpha u_{yy} = 0, \qquad \alpha = \pm 1 \tag{1.4.68}$$

and Zakharov-Kuznetsov (ZK) equation

$$u_t + (u_{xx} + u_{yy})_x + uu_x = 0. \tag{1.4.69}$$

The equation (1.4.68) derived by Kadomtsev and Petviashvili [45] describes the evolution of weakly two dimensional long water waves of small amplitude on the surface, when the wave motion is essentially one-directional with weak transverse effects along y-axis. Equation (1.4.68) is known as KP-I or KP-II equation according as $\alpha = 1$ or $\alpha = -1$ respectively.

Significant amount of work has been devoted to address the Cauchy problem associated with the KP equation, see for example [15], [38]–[43], [66], [84], [85], [88] and references therein. Here we are not going to deal with this problem. For our purpose H^1-well-posedness of the associated Cauchy problem is enough. Finally, let us record that the quantities

$$\int_{\mathbb{R}^2} u^2 \, dx \, dy, \tag{1.4.70}$$

$$\frac{1}{2}\int_{\mathbb{R}^2} \left[u_x^2 - (\partial_x^{-1} u_y)^2 - \frac{1}{3}u^3\right] dx \, dy, \tag{1.4.71}$$

are conserved by the KP-II flow which are useful to get global solution to the associated Cauchy problem in certain Sobolev spaces.

The equation (1.4.69) derived by Zakharov and Kuznetsov [89] models the propagation of nonlinear ion-acoustic waves in magnetized plasma. Much effort has been devoted to study several properties of these models, see for example [26], [6] and references therein. In particular, the well-posedness issue for the IVP associated to (1.4.69) has also been studied extensively in recent literature. Using the method developed by Kenig, Ponce and Vega [57] to show local well-posedness for the IVP associated with the KdV equation in $H^s(\mathbb{R})$, $s > 3/4$, Faminskii [26] proved the local well-posedness for the IVP associated to (1.4.69) when the given data is in $H^m(\mathbb{R}^2)$, $m \geq 1$, integer. He also proved the global well-posedness in the same space using the conserved quantities

$$\int_{\mathbb{R}^2} u^2(t) \, dxdy = \int_{\mathbb{R}^2} u_0^2 \, dxdy \tag{1.4.72}$$

and

$$\int_{\mathbb{R}^2} (u_x^2 + u_y^2 - \frac{1}{3}u^3)(t) \, dxdy = \int_{\mathbb{R}^2} ({u_0}_x^2 + u_{0y}^2 - \frac{1}{3}u_0^3) \, dxdy, \tag{1.4.73}$$

satisfied by the flow of (1.4.69).

In this work, we are concerned about the following question: If a sufficiently smooth real valued solution $u = u(x, y, t)$ to the IVP associated to (1.4.68) or (1.4.69) is supported compactly on a certain time interval, is it true that $u \equiv 0$? In some sense, it is a weak version of the unique continuation property (UCP) which is defined as follows:

Definition 3. *If a solution u to certain evolution equation vanishes on some non-empty open set Ω_1 of Ω then it vanishes in the horizontal component of Ω_1 in Ω, where Ω is the domain of the evolution operator under consideration.*

A pioneer work in this direction is due to Carleman [19]. Carleman's method was based on the weighted estimates for the associated solutions. Later, Carleman's method was improved and extended to address the UCP for parabolic and hyperbolic operators (see [35] and [65]). As far as we know the first work dealing with the UCP for a general class of dispersive equations in one space dimension is due to Saut and Scheurer [75]. Carleman type estimates were the main tools used by them. In particular, the class considered in [75] includes the KdV equation. Also, D. Tataru [87] proved the UCP for Schrödinger equation by deriving some Carleman type estimates. Further, Isakov [44] considered a large class of evolution equations with nonhomogeneous principal part and proved the UCP. Later, Zhang [93] proved the UCP for the KdV and modified KdV (mKdV) equations using inverse scattering theory and Miura's transformation. This slightly stronger result implies the UCP for the KdV equation obtained in [75]. To prove this result, Zhang [93] introduced some decay condition to the solution and exploited the fact that the KdV and mKdV equations are integrable. Bourgain [16] introduced a new approach to address a wider class of evolution equations using complex variables techniques. The method introduced in [16] is more general and can also be applied to models in higher spatial dimensions. Recently, Kenig, Ponce and Vega [49], using Carleman's type estimate and the result due to Saut and Scheurer [75] proved that; if a sufficiently smooth solution u of the generalized KdV equation is supported in $(-\infty, b)$ or in (a, ∞) at two different instants of time then $u \equiv 0$. The exponential decay property of the solution is essential in the argument employed in [49]. Quite recently, Carvajal and Panthee [21] extended the argument introduced in [16] and [49] to prove the UCP for Hasegawa-Kodama equation which is a mixed equation of type KdV and Schrödinger. Also there are recent works due to Iório in [36] and [37] dealing with the UCP for equations of Benjamin-Ono type and Kenig, Ponce Vega [48] for nonlinear Schrödinger equation.

Here we are going to generalize the scheme in [16] to address a bi-dimensional (spatial) model and provide an affirmative answer to the question posed above.

First we provide an affirmative answer to the question posed above for the ZK model. Although, employing this method, one can deduce UCP for the linear problem almost immediately, the same is not so simple for the nonlinear problem and is quite involved. The symbol associated with the linear operator and the appropriate choice of the parameter play important role in the approach we used. The positive result obtained for the ZK model motivates one to think for the similar result for the KP equation. Unlike ZK model, there is singularity in the associated symbol of the linear KP operator. So, one needs to handle the analysis with utmost care. The structure of the associated symbol has also influenced a lot in the well-posedness results for the Cauchy problem for the KP equation. In this sense, the KP-II equation is much better than the KP-I equation, see for example [15], [38]–[43], [66], [84], [85] and [88]. The structure of the associated symbol has also affected our result on UCP for the KP equation. Here, we are able to handle only the KP-II equation by choosing appropriate parameters, see Remark 5.3 below. Therefore, from here onwards, we concentrate our work on KP-II equation (i.e., the IVP (5.2.48) with $\alpha = -1$) and obtain UCP for it.

We organize this work as follows. Chapter 2 contains the results concerning the Cauchy problem for the IVP (1.3.57). Chapter 3 deals with the global solution to the system of the mKdV equations. The local and global well-posedness issues for the system of the cKdV equations are treated in Chapter 4. Finally, in Chapter 5 we establish the unique continuation property for the ZK and KP-II equations.

Chapter 2

Coupled System of the KdV Equations

2.1 Introduction

This chapter is devoted to investigate the well-posedness issues associated to the IVP the Gear and Gimshaw

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_2 (uv)_x = 0, \qquad x,\, t \in \mathbb{R}, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0, \\ u(x,0) = \phi(x), \\ v(x,0) = \psi(x), \end{cases} \tag{2.1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are real valued functions and a_1, a_2, a_3, b_1, b_2 are real constants with b_1, b_2 positive. Recall that the IVP (2.1.1) is the particular case of the IVP (1.3.57) for $r = 0$.

The function space where we are going to find solution to the IVP (2.1.1) is the Hilbert space $X_{s,b}$ which is a completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm defined in (1.2.46).

Now we state the main results of this chapter. The first result is concerned about the local well-posedness for the IVP (2.1.1) and reads as follows.

Theorem 2.1. *For any* $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$ *and* $b \in (1/2, 1)$, *there exist* $T = T(\|\phi\|_{H^s}, \|\psi\|_{H^s})$ *and a unique solution of (2.1.1) in the time interval* $[-T, T]$ *satisfying*

$$u, v \in C([-T,T]; H^s(\mathbb{R})), \tag{2.1.2}$$

$$u, v \in X_{s,b} \subseteq L^p_{x,loc}(\mathbb{R}; L^2_t(\mathbb{R})), \quad for\ 1 \leq p \leq \infty, \tag{2.1.3}$$

$$(u^2)_x, (v^2)_x \in X_{s,b-1}, \tag{2.1.4}$$

and

$$u_t, v_t \in X_{s-3,b-1}. \tag{2.1.5}$$

Moreover, given $T' \in (0,T)$, the map $(\phi, \psi) \mapsto (u(t), v(t))$ is smooth from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $C([-T', T']; H^s(\mathbb{R})) \times C([-T', T']; H^s(\mathbb{R}))$.

Our next theorem deals with the global well-posedness for the IVP (2.1.1). More precisely, we prove the following result.

Theorem 2.2. *The initial value problem (2.1.1) is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/10$ in the case when $a_3 = 0$ and $b_1 = b_2$.*

The final result of this chapter is concerned about the ill-posedness for the IVP (2.1.1). In fact, we prove the following theorem showing that the local result given by Theorem 2.1 is sharp.

Theorem 2.3. *Let $s < -3/4$, then there is no $T > 0$ such that the flow-map*

$$(\phi, \psi) \mapsto (u(t), v(t)), \qquad t \in (0, T]$$

be C^2 Frechet-differentiable at the origin from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R}))$.

2.2 Reduction of the Problem and Preliminary Estimates

In this section we decouple the dispersive terms in the system (2.1.1). Also we recall some estimates that will be useful in the proof of Theorem 2.1.

If $a_3 = 0$ there is no coupling in the dispersive terms. So we suppose $a_3 \neq 0$. As mentioned above, we are interested to decouple the dispersive terms in the system (2.1.1). For this, let $a_3^2 b_2 \neq 1$ and define,

$$\lambda = \left\{ \left(1 - \frac{1}{b_1}\right)^2 + \frac{4 b_2 a_3^2}{b_1} \right\}^{1/2} > 0 \quad \text{and} \quad \alpha_\pm = \frac{1}{2}\left(1 + \frac{1}{b_1} \pm \lambda\right).$$

Our assumption $a_3^2 b_2 \neq 1$ guarantees that $\alpha_\pm \neq 0$. Thus we can write the system (2.1.1) in a matrix form and then diagonalize the matrix of coefficients corresponding to the dispersive terms as in [74]. After that we make the change of scale

$$u(x,t) = \tilde{u}(\alpha_+^{-1/3}x, t) \qquad \text{and} \qquad v(x,t) = \tilde{v}(\alpha_-^{-1/3}x, t).$$

Then we obtain the system of equations

$$\begin{cases} \tilde{u}_t + \tilde{u}_{xxx} + a\tilde{u}\tilde{u}_x + b\tilde{v}\tilde{v}_x + c(\tilde{u}\tilde{v})_x = 0, \\ \tilde{v}_t + \tilde{v}_{xxx} + \tilde{a}\tilde{u}\tilde{u}_x + \tilde{b}\tilde{v}\tilde{v}_x + \tilde{c}(\tilde{u}\tilde{v})_x = 0, \\ \tilde{u}(x,0) = \tilde{\phi}(x), \\ \tilde{v}(x,0) = \tilde{\psi}(x), \end{cases} \tag{2.2.1}$$

where a, b, c and $\tilde{a}, \tilde{b}, \tilde{c}$ are constants.

Remark 2.1. *Notice that the nonlinear terms involving the functions $\tilde{u}$ and $\tilde{v}$ are not evaluated at the same point. Therefore those terms are not local anymore. Hence we should be careful in making the estimates. In the existing literature, see for instance [74] and [5], this feature of the nonlinear terms was not pointed out which may lead to wrong conclusions.*

Remark 2.2. *Due to the previous remark, in Proposition 2.1 below, we need to estimate terms of the form $\partial_x(u(Ax,t)v(Bx,t))$ or more generally $\partial_x(u(Ax+C,t)v(Bx+D,t))$. It is not difficult to prove the same inequality since the only changes coming out are from the contributions given by the constants A, B, C and D.*

The system (2.2.1) has a pair of KdV equations coupled only in the nonlinear terms. It is enough to prove local well-posedness for system (2.2.1) because the results for the IVP (2.1.1) can be obtained in the obvious way.

Hence, our interest is to solve the system (2.2.1) for initial data $(\tilde{\phi}, \tilde{\psi})$ in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$. For this we use the Fourier transform restriction space $X_{s,b}$ discussed above. For the sake of simplicity, from now onwards we will drop '~' and use the notation u, v, ϕ, ψ in the system (2.2.1).

Using Duhamel's principle, we study the following system of integral equations equivalent to the system (2.2.1),

$$\begin{cases} u(t) = U(t)\phi - \int_0^t U(t-t')F(u,v,u_x,v_x)(t')\,dt', \\ v(t) = U(t)\psi - \int_0^t U(t-t')G(u,v,u_x,v_x)(t')\,dt', \end{cases} \tag{2.2.2}$$

where $U(t) = e^{-t\partial_x^3}$ is the unitary group associated with the linear problem and F and G are respective nonlinearities.

To find a local solution to (2.2.1) we can replace (2.2.2) with the following system of integral equations,

$$\begin{cases} u(t) = \psi_1(t)U(t)\phi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')F(u,v,u_x,v_x)(t')\,dt', \\ v(t) = \psi_1(t)U(t)\psi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')G(u,v,u_x,v_x)(t')\,dt', \end{cases} \tag{2.2.3}$$

where $\psi_1 \in C_0^\infty(\mathbb{R})$, $0 \le \psi_1 \le 1$ is a cut-off function given by,

$$\psi_1 = \begin{cases} 1, & |t| < 1 \\ 0, & |t| \ge 2 \end{cases}$$

and $\psi_\delta = \psi_1(t/\delta),\ 0 < \delta \le 1$.

Now, let us recall some estimates which will be used to prove the local well-posedness result.

Lemma 2.1. *Let $s \in \mathbb{R}$, $b', b \in (1/2, 1)$ with $b' < b$ and $\delta \in (0,1)$; then we have,*

$$\|\psi_\delta(t)U(t)\phi\|_{X_{s,b}} \le C\,\delta^{\frac{(1-2b)}{2}}\|\phi\|_{H^s}, \tag{2.2.4}$$

$$\|\psi_\delta F\|_{X_{s,b-1}} \le C\delta^{\frac{b-b'}{8(1-b')}}\|F\|_{X_{s,b'-1}}, \tag{2.2.5}$$

$$\left\|\psi_\delta(t)\int_0^t U(t-t')F(t')\,dt'\right\|_{X_{s,b}} \le C\,\delta^{\frac{(1-2b)}{2}}\|F\|_{X_{s,b-1}} \tag{2.2.6}$$

and

$$\left\|\psi_\delta(t)\int_0^t U(t-t')F(t')\,dt'\right\|_{H^s} \le C\,\delta^{\frac{(1-2b)}{2}}\|F\|_{X_{s,b-1}}. \tag{2.2.7}$$

Proposition 2.1. *Let $s > -3/4$, then there exists $1/2 < b < 1$ such that the following bilinear estimate holds,*

$$\|(uv)_x\|_{X_{s,b-1}} \leq C\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}. \tag{2.2.8}$$

The proof of Lemma 2.1 can be found in ([53], [51]) and that of Proposition 2.1 in [51], so we skip the details.

2.3 Local Well-posedness Result

In this section we supply the proof of Theorem 2.1, the local well-posedness result for the IVP (2.1.1).

Proof of Theorem 2.1: We consider the following function space where we seek a solution to the IVP (2.2.1). For given $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ and $b > 1/2$, let us define,

$$\mathcal{H}_{MN} := \{(u, v) \in X_{s,b} \times X_{s,b} : \|u\|_{X_{s,b}} \leq M,\ \|v\|_{X_{s,b}} \leq N\},$$

where $M = 2C_0\|\phi\|_{H^s}$ and $N = 2C_0\|\psi\|_{H^s}$. Then $\mathcal{H}_{MN}$ is a complete metric space with norm,

$$|||(u, v)|||_{\mathcal{H}_{MN}} := \|u\|_{X_{s,b}} + \|v\|_{X_{s,b}}.$$

Without loss of generality, we may assume that $M > 1$ and $N > 1$. For $(u, v) \in \mathcal{H}_{MN}$, let us define the maps,

$$\begin{cases} \Phi_\phi[u, v] = \psi_1(t)U(t)\phi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')F(u, v, u_x, v_x)(t')\,dt' \\ \Psi_\psi[u, v] = \psi_1(t)U(t)\psi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')G(u, v, u_x, v_x)(t')\,dt'. \end{cases} \tag{2.3.1}$$

We prove that $\Phi \times \Psi$ maps $\mathcal{H}_{MN}$ into $\mathcal{H}_{MN}$ and is a contraction.

Using (2.2.4) and (2.2.6) we get from (2.3.1),

$$\begin{cases} \|\Phi[u, v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C\|\psi_\delta F(u, v, u_x, v_x)\|_{X_{s,b-1}} \\ \|\Psi[u, v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C\|\psi_\delta G(u, v, u_x, v_x)\|_{X_{s,b-1}}. \end{cases} \tag{2.3.2}$$

Now, using (2.2.5) we get from (2.3.2) for $b' < b$ and $\theta = \frac{b-b'}{8(1-b')}$,

$$\begin{cases} \|\Phi[u,v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C\delta^\theta \|F(u,v,u_x,v_x)\|_{X_{s,b'-1}} \\ \|\Psi[u,v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C\delta^\theta \|G(u,v,u_x,v_x)\|_{X_{s,b'-1}}. \end{cases} \tag{2.3.3}$$

Using the bilinear estimate (2.2.8), the estimate (2.3.3) yields,

$$\begin{cases} \|\Phi[u,v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C_1\delta^\theta \{\|u\|^2_{X_{s,b}} + \|v\|^2_{X_{s,b}} + \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\} \\ \|\Psi[u,v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C_2\delta^\theta \{\|u\|^2_{X_{s,b}} + \|v\|^2_{X_{s,b}} + \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\}. \end{cases} \tag{2.3.4}$$

As $(u,v) \in \mathcal{H}_{MN}$, with our choice of M and N we get from (2.3.4),

$$\begin{cases} \|\Phi[u,v]\|_{X_{s,b}} \leq \frac{M}{2} + C_1\delta^\theta\{M^2 + N^2 + MN\} \\ \|\Psi[u,v]\|_{X_{s,b}} \leq \frac{N}{2} + C_2\delta^\theta\{M^2 + N^2 + MN\}. \end{cases} \tag{2.3.5}$$

If we choose δ such that

$$\delta^\theta \leq \left(2\max\{C_1, C_2\}(M+N)^2\right)^{-1},$$

then we get from (2.3.5),

$$\|\Phi[u,v]\|_{X_{s,b}} \leq M \quad \text{and} \quad \|\Psi[u,v]\|_{X_{s,b}} \leq N.$$

Therefore,

$$(\Phi[u,v], \Psi[u,v]) \in \mathcal{H}_{MN}.$$

Now, we need to show that $\Phi \times \Psi : (u,v) \mapsto (\Phi[u,v], \Psi[u,v])$ is a contraction. For this, let $(u,v), (u_1,v_1) \in \mathcal{H}_{MN}$, then as above using Lemma 2.1 and Proposition 2.1 we get,

$$\begin{cases} \|\Phi[u,v] - \Phi[u_1,v_1]\|_{X_{s,b}} \leq C_1\delta^\theta(M+N)\big[\|u-u_1\|_{X_{s,b}} + \|v-v_1\|_{X_{s,b}}\big] \\ \|\Psi[u,v] - \Psi[u_1,v_1]\|_{X_{s,b}} \leq C_2\delta^\theta(M+N)\big[\|u-u_1\|_{X_{s,b}} + \|v-v_1\|_{X_{s,b}}\big]. \end{cases} \tag{2.3.6}$$

If we choose δ such that

$$\delta^\theta \leq \left(4\max\{C_1, C_2\}(M+N)^2\right)^{-1},$$

then (2.3.6) yields,

$$\begin{cases} \|\Phi[u,v]-\Phi[u_1,v_1]\|_{X_{s,b}} \leq \frac{1}{4}[\|u-u_1\|_{X_{s,b}}+\|v-v_1\|_{X_{s,b}}] \\ \|\Psi[u,v]-\Psi[u_1,v_1]\|_{X_{s,b}} \leq \frac{1}{4}[\|u-u_1\|_{X_{s,b}}+\|v-v_1\|_{X_{s,b}}]. \end{cases} \tag{2.3.7}$$

Therefore the map $\Phi \times \Psi$ is a contraction and we obtain a unique fixed point (u, v) which solves the IVP (2.2.1) for $t \in [-T, T]$ with $T \leq \delta$. The remainder of the proof follows a standard argument so we skip it. Just to be precise, the smoothness of the flow-map follows by using Implicit Function Theorem. □

2.4 Global Well-posedness Result

This section is devoted to extend the local solution obtained in the previous section to the global one. Using usual conservation laws we have global solution in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 0$. So, we suppose $s < 0$ throughout this section. Our aim here is to derive an *almost conserved quantity* and use it to prove Theorem 2.2. For this, let us define the Fourier multiplier operator I by,

$$\widehat{Iu}(\xi) = m(\xi)\hat{u}(\xi),$$

where $m(\xi)$ is a smooth and monotone function given by

$$m(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s}|\xi|^s, & |\xi| \geq 2N, \end{cases}$$

with $N \gg 1$ to be fixed later.

Note that, I is the identity operator in low frequencies, $\{\xi : |\xi| < N\}$, and simply an integral operator in high frequencies. In general, it commutes with differential operators and satisfies the following property.

Lemma 2.2. *Let $-3/4 < s < 0$ and $N \gg 1$. Then the operator I maps $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$ and*

$$\|If\|_{L^2(\mathbb{R})} \lesssim N^{-s}\|f\|_{H^s(\mathbb{R})}. \tag{2.4.1}$$

Proof.

$$\begin{aligned}
\|If\|_{L^2}^2 &= \|\widehat{If}\|_{L^2}^2 = \|m(\cdot)\hat{f}\|_{L^2}^2 \\
&= \int\limits_{|\xi|<N} |\hat{f}(\xi)|^2\,d\xi + \int\limits_{N\le|\xi|\le 2N} |m(\xi)|^2|\hat{f}(\xi)|^2\,d\xi + \int\limits_{|\xi|>2N} N^{-2s}|\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi \\
&\le CN^{-2s}\|f\|_{H^s}^2 + N^{-2s}\int_{|\xi|>2N}(1+\xi^2)^s|\hat{f}(\xi)|^2(1+\frac{1}{\xi^2})^{-s}\,d\xi \\
&\le CN^{-2s}\|f\|_{H^s}^2.
\end{aligned}$$

□

As discussed in the introduction let us consider the IVP (2.1.1) with $a_3 = 0$ and $b_1 = b_2$, that is,

$$\begin{cases}
u_t + u_{xxx} + uu_x + a_1vv_x + a_2(uv)_x = 0,\\
b_1v_t + v_{xxx} + vv_x + b_2a_2uu_x + b_2a_1(uv)_x = 0,\\
u(x,0) = \phi(x),\\
v(x,0) = \psi(x).
\end{cases} \tag{2.4.2}$$

After introducing the multiplier operator I, we have the following variant of the local well-posedness for the IVP (2.4.2).

Theorem 2.4. *For any $(\phi,\psi) \in H^s(\mathbb{R})\times H^s(\mathbb{R})$, $s > -3/4$, the IVP (2.4.2) is locally well-posed in the Banach space $I^{-1}L^2\times I^{-1}L^2 = \{(\phi,\psi)\in H^s\times H^s,\ \text{with norm } \|I\phi\|_{L^2}+\|I\psi\|_{L^2}\}$ with the existence lifetime satisfying,*

$$\delta \gtrsim (\|I\phi\|_{L^2}^2 + \|I\psi\|_{L^2}^2)^{-\theta}, \qquad \theta > 0. \tag{2.4.3}$$

Moreover,

$$\begin{cases}
\|\psi_\delta Iu\|_{X_{0,b}} \le C\|I\phi\|_{L^2}\\
\|\psi_\delta Iv\|_{X_{0,b}} \le C\|I\psi\|_{L^2}.
\end{cases} \tag{2.4.4}$$

The proof of this theorem is not difficult and follows by using the same procedure used to prove the local well-posedness for the IVP (2.2.1) (see the Proof of Theorem 2.1) once we have the bilinear estimate

$$\|\partial_x I(uv)\|_{X_{0,-\frac{1}{2}+}} \leq C\|Iu\|_{X_{0,\frac{1}{2}+}}\|Iv\|_{X_{0,\frac{1}{2}+}}. \tag{2.4.5}$$

The proof of the bilinear estimate (2.4.5) is easy and follows by using the usual bilinear estimate (2.2.8) due to Kenig, Ponce and Vega [51] combined with the following extra smoothing bilinear estimate whose proof is given in Colliander, Keel, Staffilani, Takaoka and Tao [25].

Proposition 2.2. *The bilinear estimate*

$$\|\partial_x\{(IuIv - I(uv)\}\|_{X^\delta_{0,-\frac{1}{2}+}} \leq CN^{-\frac{3}{4}+}\|Iu\|_{X^\delta_{0,\frac{1}{2}+}}\|Iv\|_{X^\delta_{0,\frac{1}{2}+}}, \tag{2.4.6}$$

holds.

Now we proceed to introduce the almost conserved quantity. Using the Fundamental Theorem of Calculus, the equation and integration by parts we get,

$$\begin{aligned}
\|Iu(\delta)\|_{L^2}^2 =&\|Iu(0)\|_{L^2}^2 + \int_0^\delta \frac{d}{dt}(Iu(t), Iu(t))\,dt\\
=&\|Iu(0)\|_{L^2}^2 + 2\int_0^\delta \Big(\frac{d}{dt}Iu(t), Iu(t)\Big)\,dt\\
=&\|Iu(0)\|_{L^2}^2 + 2\int_0^\delta (I(-u_{xxx} - uu_x - a_1vv_x - a_2(uv)_x), Iu(t))\;dt\\
=&\|Iu(0)\|_{L^2}^2 + 2\int_0^\delta (I(-uu_x - a_1vv_x - a_2(uv)_x), Iu(t))\;dt\\
=&\|Iu(0)\|_{L^2}^2 + R_1(\delta),
\end{aligned} \tag{2.4.7}$$

where

$$R_1(\delta) = \int_0^\delta\int_{\mathbb{R}} \partial_x(-Iu^2 - a_1Iv^2 - 2a_2I(uv))Iu\,dxdt. \tag{2.4.8}$$

Similarly,

$$\|Iv(\delta)\|_{L^2}^2 = \|Iv(0)\|_{L^2}^2 + R_2(\delta), \tag{2.4.9}$$

where

$$R_2(\delta) = \int_0^\delta\int_{\mathbb{R}} \partial_x(-\frac{1}{b_1}Iv^2 - \frac{b_2a_2}{b_1}Iu^2 - \frac{2b_2a_1}{b_1}I(uv))Iv\,dxdt. \tag{2.4.10}$$

Let us define $R(\delta) := R_1(\delta) + R_2(\delta)$, so that we have from (2.4.7) and (2.4.9),

$$\|Iu(\delta)\|_{L^2}^2 + \|Iv(\delta)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + \|Iv(0)\|_{L^2}^2 + R(\delta). \tag{2.4.11}$$

We will obtain the so called almost conserved quantity from (2.4.11) by treating $R(\delta)$ as an error term. In what follows we prove a cancellation property which plays a vital role in our analysis.

Lemma 2.3. *The following cancellations hold.*

$$\int_0^\delta \int_{\mathbb{R}} \partial_x (Iu)^2 Iu \, dxdt = 0 = \int_0^\delta \int_{\mathbb{R}} \partial_x (Iv)^2 Iv \, dxdt \tag{2.4.12}$$

and

$$a_1 b_1 I_1 + 2a_2 b_1 I_2 + b_2 a_2 I_3 + 2b_2 a_1 I_4 = 0, \quad \text{if} \quad b_1 = b_2, \tag{2.4.13}$$

where

$$\begin{aligned} I_1 &= \int_0^\delta \int_{\mathbb{R}} \partial_x (Iv)^2 Iu \, dxdt, & I_2 &= \int_0^\delta \int_{\mathbb{R}} \partial_x (IuIv) Iu \, dxdt, \\ I_3 &= \int_0^\delta \int_{\mathbb{R}} \partial_x (Iu)^2 Iv \, dxdt \quad \text{and} & I_4 &= \int_0^\delta \int_{\mathbb{R}} \partial_x (IuIv) Iv \, dxdt. \end{aligned}$$

Proof. The proof of (2.4.12) is trivial and (2.4.13) follows by using integration by parts. In fact, for $b_1 = b_2$,

$$\begin{aligned} a_1 I_1 + 2a_2 I_2 + a_2 I_3 + 2a_1 I_4 = \\ = a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x (Iv)^2 Iu \, dxdt + 2a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x (IuIv) Iu \, dxdt \\ + a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x (Iu)^2 Iv \, dxdt + 2a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x (IuIv) Iv \, dxdt \\ = - a_1 \int_0^\delta \int_{\mathbb{R}} (Iv)^2 \partial_x Iu \, dxdt - a_2 \int_0^\delta \int_{\mathbb{R}} Iv \partial_x (Iu)^2 \, dxdt \\ - a_2 \int_0^\delta \int_{\mathbb{R}} (Iu)^2 \partial_x Iv \, dxdt - a_1 \int_0^\delta \int_{\mathbb{R}} Iu \partial_x (Iv)^2 \, dxdt \\ = - a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x [Iu (Iv)^2] \, dxdt - a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x [Iv (Iu)^2] \, dxdt \\ = 0. \end{aligned}$$

□

Remark 2.3. *Note that, it is here in Lemma 2.3, where the restriction $b_1 = b_2$ on the coefficients appears. From here onwards we will use this restriction on the coefficients.*

Using Lemma 2.3, $R(\delta)$ can be written as,

$$
\begin{aligned}
&R(\delta) = \\
&\int_0^\delta \int_{\mathbb{R}} \partial_x\{(Iu)^2 - I(u^2)\} Iu\, dxdt + \frac{1}{b_1}\int_0^\delta \int_{\mathbb{R}} \partial_x\{(Iv)^2 - (Iv^2)\} Iv\, dxdt \\
&+ a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x\{(Iv)^2 - I(v^2)\} Iu\, dxdt + a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x\{(Iu)^2 - (Iu^2)\} Iv\, dxdt \\
&+ 2a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x\{IuIv - I(uv)\} Iu\, dxdt + 2a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x\{IuIv - I(uv)\} Iv\, dxdt.
\end{aligned}
\tag{2.4.14}
$$

Using Plancherel identity and Cauchy-Schwarz inequality as in [79] we get,

$$
\begin{aligned}
|R(\delta)| \leq C\Big\{ &\|\partial_x\{(IuIv - I(uv)\}\|_{X^\delta_{0,-\frac{1}{2}+}} \big(\|Iu\|_{X^\delta_{0,\frac{1}{2}+}} + \|Iv\|_{X^\delta_{0,\frac{1}{2}+}}\big) \\
&+ \left(\|\partial_x\{(Iu)^2 - I(u^2)\}\|_{X^\delta_{0,-\frac{1}{2}+}} + \|\partial_x\{(Iv)^2 - I(v^2)\}\|_{X^\delta_{0,-\frac{1}{2}+}}\right) \|Iu\|_{X^\delta_{0,\frac{1}{2}+}} \\
&+ \left(\|\partial_x\{(Iv)^2 - I(v^2)\}\|_{X^\delta_{0,-\frac{1}{2}+}} + \|\partial_x\{(Iu)^2 - I(u^2)\}\|_{X^\delta_{0,-\frac{1}{2}+}}\right) \|Iv\|_{X^\delta_{0,\frac{1}{2}+}} \Big\}.
\end{aligned}
\tag{2.4.15}
$$

Now, using (2.4.15) and Proposition 2.2 the identity (2.4.11) yields the following almost conservation law,

$$
\begin{aligned}
\|Iu(\delta)\|^2_{L^2} + \|Iv(\delta)\|^2_{L^2} \leq{}& \|Iu(0)\|^2_{L^2} + \|Iv(0)\|^2_{L^2} \\
&+ CN^{-\frac{3}{4}+}\Big\{ \|Iu\|^3_{X^\delta_{0,\frac{1}{2}+}} + \|Iu\|^2_{X^\delta_{0,\frac{1}{2}+}} \|Iv\|_{X^\delta_{0,\frac{1}{2}+}} \\
&+ \|Iv\|^2_{X^\delta_{0,\frac{1}{2}+}} \|Iu\|_{X^\delta_{0,\frac{1}{2}+}} + \|Iv\|^3_{X^\delta_{0,\frac{1}{2}+}} \Big\}.
\end{aligned}
\tag{2.4.16}
$$

Now we are in position to prove the global well-posedness result.

Proof of Theorem 2.2: To prove the theorem it is enough to show that the local solution to the IVP (2.4.2) can be extended to $[0, T]$ for arbitrary $T > 0$. To make the analysis easy we use the scaling introduced in the introduction. That is, if (u, v) solves the IVP (2.4.2) with initial data (ϕ, ψ) then for $1 > \lambda > 0$ so does (u^λ, v^λ); where $u^\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^3 t)$, $v^\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^3 t)$; with initial data $(\phi^\lambda, \psi^\lambda)$ given by $\phi^\lambda(x) = \lambda^2\phi(\lambda x)$, $\psi^\lambda(x) = \lambda^2\psi(\lambda x)$. Observe that, (u, v) exists in $[0, T]$ if and only if (u^λ, v^λ) exists in $[0, T/\lambda^3]$. So we are interested in extending (u^λ, v^λ) to $[0, T/\lambda^3]$.

Using Lemma 2.2 we have,

$$\begin{cases} \|I\phi^\lambda\|_{L^2} \leq C\lambda^{\frac{3}{2}+s} N^{-s} \|\phi\|_{H^s}, \\ \|I\psi^\lambda\|_{L^2} \leq C\lambda^{\frac{3}{2}+s} N^{-s} \|\psi\|_{H^s}. \end{cases} \tag{2.4.17}$$

$N = N(T)$ will be selected later, but let us choose $\lambda = \lambda(N)$ right now by requiring that,

$$\begin{cases} C\,\lambda^{\frac{3}{2}+s} N^{-s} \|\phi\|_{H^s} = \sqrt{\frac{\epsilon_0}{2}} \ll 1, \\ C\,\lambda^{\frac{3}{2}+s} N^{-s} \|\psi\|_{H^s} = \sqrt{\frac{\epsilon_0}{2}} \ll 1. \end{cases} \tag{2.4.18}$$

From (2.4.18) we get, $\lambda \sim N^{\frac{2s}{3+2s}}$ and using (2.4.18) in (2.4.17) we get,

$$\begin{cases} \|I\phi^\lambda\|_{L^2}^2 \leq \frac{\epsilon_0}{2} \ll 1 \\ \|I\psi^\lambda\|_{L^2}^2 \leq \frac{\epsilon_0}{2} \ll 1. \end{cases} \tag{2.4.19}$$

Therefore, if we choose ϵ_0 arbitrarily small then from Theorem 2.4 we see that the IVP (2.4.2) is well-posed for all $t \in [0, 1]$.

Now, using the almost conserved quantity (2.4.16), the identity (2.4.19) and Theorem 2.4, we get,

$$\begin{aligned} \|Iu^\lambda(1)\|_{L^2}^2 + \|Iv^\lambda(1)\|_{L^2}^2 &\leq \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} + CN^{-\frac{3}{4}+}\{3\frac{\epsilon_0}{2}(\frac{\epsilon_0}{2})^{1/2}\} \\ &\leq \epsilon_0 + CN^{-\frac{3}{4}+}\epsilon_0. \end{aligned} \tag{2.4.20}$$

So, we can iterate this process $C^{-1}N^{\frac{3}{4}-}$ times before doubling $\|Iu^\lambda(t)\|_{L^2}^2 + \|Iv^\lambda(t)\|_{L^2}^2$. By this process we can extend the solution to the time interval $[0,\, C^{-1}N^{\frac{3}{4}-}]$ by taking $C^{-1}N^{\frac{3}{4}-}$ time steps of size $O(1)$. As we are interested in extending the solution to the time interval

$[0,\, T/\lambda^3]$, let us select $N = N(T)$ such that, $C^{-1}N^{\frac{3}{4}-} \geq T/\lambda^3$. That is,

$$N^{\frac{3}{4}-} \geq C\frac{T}{\lambda^3} \sim TN^{\frac{-6s}{3+2s}}.$$

Therefore for large N, the existence interval will be arbitrarily large if we choose s such that $s > -3/10$. This completes the proof of the theorem. $\square$

2.5 Ill-posedness Result

As in the local well-posedness result, we consider the system (2.2.1). Note that we have dropped '~' and retained the notation u, v, ϕ and ψ. Let $W = (u, v)^T$, then the IVP (2.2.1) can be written as,

$$\begin{cases} W_t + W_{xxx} + B(W)W_x = 0, \\ W(x,0) = W_0(x), \end{cases} \tag{2.5.1}$$

where,

$$B(W) = \begin{pmatrix} au + dv & cu + bv \\ \tilde{a}u + \tilde{d}v & \tilde{c}u + \tilde{b}v \end{pmatrix}.$$

For fixed $\Phi \in \dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$, consider the solution $W = W^\delta$ of the IVP

$$\begin{cases} W_t + W_{xxx} + B(W)W_x = 0, \\ W(x,0) = \delta\Phi(x), \qquad \delta \in \mathbb{R}. \end{cases} \tag{2.5.2}$$

We will show that the flow-map $\delta\Phi \mapsto W^\delta(x,t)$ fails to be C^2 at the origin when $s < -3/4$. More precisely, we prove the following theorem which in turn implies Theorem 2.3.

Theorem 2.5. *Let $s < -3/4$, then there is no $T > 0$ such that the flow-map*

$$\delta\Phi \mapsto W^\delta(t), \qquad t \in (0,T]$$

be C^2 Frechet-differentiable at the origin from $\dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$ to $C([0,T]; \dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R}))$.

Proof. We prove it by contradiction. Suppose that the flow-map be C^2 differentiable at the origin. Using Duhamel's formula we have from (2.5.2),

$$W^\delta(x,t) = \delta U(t)\Phi(x) - \int_0^t U(t-t')B(W^\delta(x,t'))W_x^\delta(x,t')\,dt', \tag{2.5.3}$$

where $U(t)$ is the unitary group associated to the linear problem. Differentiating (2.5.3) with respect to δ we get,

$$\frac{\partial W^\delta(x,t)}{\partial\delta}\Big|_{\delta=0} = U(t)\Phi(x) := W_1(x,t), \tag{2.5.4}$$

$$\frac{\partial^2 W^\delta(x,t)}{\partial\delta^2}\Big|_{\delta=0} = -2\int_0^t U(t-t')B(W_1(x,t'))W_{1x}(x,t')\,dt' := W_2(x,t). \tag{2.5.5}$$

Our assumption of the C^2 regularity of the flow-map at the origin implies that,

$$\|W_2(\cdot,t)\|_{\dot H^s\times\dot H^s} \le C\,\|\Phi\|^2_{\dot H^s\times\dot H^s}. \tag{2.5.6}$$

Now, we look for $\Phi \in \dot H^s(\mathbb{R})\times\dot H^s(\mathbb{R})$ so that (2.5.6) fails to hold whenever $s < -3/4$. For this, let $I_1 := [-N, -N+\alpha]$, $I_2 := [N+\alpha, N+2\alpha]$ with $N \gg 1$ and $\alpha \ll 1$, define ϕ by the formula

$$\hat\phi(\xi) = \alpha^{-\frac{1}{2}}N^{-s}\big\{\chi_{I_1}(\xi) + \chi_{I_2}(\xi)\big\}, \tag{2.5.7}$$

and take $\Phi = (\phi,\phi)^T$.

It is easy to see that

$$\|\Phi\|_{\dot H^s\times\dot H^s} \sim 1. \tag{2.5.8}$$

Now we proceed to calculate $\|W_2(\cdot,t)\|_{\dot H^s\times\dot H^s}$. For this, let us first calculate $W_1(x,t)$ and $W_2(x,t)$.

From (2.5.4) we have ,

$$\widehat{W_1}^{(x)}(\xi,t) = e^{it\xi^3}\hat\Phi(\xi).$$

Therefore,

$$W_1(x,t) \sim \alpha^{-\frac{1}{2}} N^{-s} \begin{pmatrix} \int_{\xi \in I_1 \cup I_2} e^{ix\xi + it\xi^3}\, d\xi \\ \int_{\xi \in I_1 \cup I_2} e^{ix\xi + it\xi^3}\, d\xi \end{pmatrix}.$$

From (2.5.5) we get,

$$\begin{aligned} W_2(x,t) &= -2\int_0^t U(t-t')B(W_1(x,t'))W_{1x}(x,t')\, dt' \\ &= \int_{\mathbb{R}^2} \xi e^{ix\xi + it\xi^3} \hat{\phi}(\xi - \xi_1)\hat{\phi}(\xi_1) \begin{pmatrix} a' \, \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{3\xi\xi_1(\xi-\xi_1)} \\ b' \, \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{3\xi\xi_1(\xi-\xi_1)} \end{pmatrix} d\xi_1\, d\xi, \end{aligned}$$

where $a' = a + b + c + d$ and $b' = \tilde{a} + \tilde{b} + \tilde{c} + \tilde{d}$. Therefore, using (2.5.7) we get,

$$W_2(x,t) \sim \alpha^{-1} N^{-2s} \int_{\substack{\xi_1 \in I_1 \cup I_2 \\ \xi - \xi_1 \in I_1 \cup I_2}} \xi e^{ix\xi + it\xi^3} \begin{pmatrix} a'\, h(\xi, \xi_1, t) \\ b'\, h(\xi, \xi_1, t) \end{pmatrix} d\xi\, d\xi_1,$$

where $h(\xi, \xi_1, t) = \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}$. Hence, formally we have,

$$\begin{aligned} \widehat{W_2}^{(x)}(\xi, t) &\sim \alpha^{-1} N^{-2s} \xi e^{it\xi^3} \begin{pmatrix} a' \sum_{j=1}^3 \int_{A_j(\xi)} h(\xi, \xi_1, t)\, d\xi_1 \\ b' \sum_{j=1}^3 \int_{A_j(\xi)} h(\xi, \xi_1, t)\, d\xi_1 \end{pmatrix} \\ &:= \begin{pmatrix} p_1(\xi,t) + p_2(\xi,t) + p_3(\xi,t) \\ q_1(\xi,t) + q_2(\xi,t) + q_3(\xi,t) \end{pmatrix}, \end{aligned}$$

where,

$$\begin{cases} A_1(\xi) = \{\xi_1 : \xi_1 \in I_1\,,\ \xi - \xi_1 \in I_1\} \\ A_2(\xi) = \{\xi_1 : \xi_1 \in I_2\,,\ \xi - \xi_1 \in I_2\} \\ A_3(\xi) = \{\xi_1 : \xi_1 \in I_1\,,\ \xi - \xi_1 \in I_2 \quad \text{or} \quad \xi_1 \in I_2\,,\ \xi - \xi_1 \in I_1\}. \end{cases}$$

Let $\widehat{f_j}^{(x)} = p_j$ and $\widehat{g_j}^{(x)} = q_j$, $j = 1, 2, 3$, then,

$$\begin{aligned} W_2(x,t) &= \begin{pmatrix} f_1(x,t) \\ g_1(x,t) \end{pmatrix} + \begin{pmatrix} f_2(x,t) \\ g_2(x,t) \end{pmatrix} + \begin{pmatrix} f_3(x,t) \\ g_3(x,t) \end{pmatrix} \\ &:= F_1(x,t) + F_2(x,t) + F_3(x,t). \end{aligned} \tag{2.5.9}$$

Let us find an upper bound for $\dot{H}^s \times \dot{H}^s$ norm of F_1. If $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_1$ then $|\xi_1| \sim |\xi - \xi_1| \sim |\xi| \sim N$ and we get,

$$\begin{aligned}
\|F_1\|^2_{\dot{H}^s \times \dot{H}^s} &= \int_{\mathbb{R}} |\xi|^{2s}(|p_1(\xi,t)|^2 + |q_1(\xi,t)|^2)\, d\xi \\
&\sim \int_{\mathbb{R}} |\xi|^{2s}\alpha^{-2}N^{-4s}|\xi|^2(a'^2+b'^2)\Big| \int_{A_1(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\, d\xi_1\Big|^2 d\xi \\
&\leq C\alpha^{-2}N^{-2s-4}(a'^2+b'^2)\alpha^2\alpha \\
&= C\alpha N^{-2s-4}.
\end{aligned}$$

Therefore,

$$\|F_1\|^2_{\dot{H}^s \times \dot{H}^s} \leq C\alpha^{\frac{1}{2}}N^{-s-2}. \tag{2.5.10}$$

Similarly,

$$\|F_2\|^2_{\dot{H}^s \times \dot{H}^s} \leq C\alpha^{\frac{1}{2}}N^{-s-2}. \tag{2.5.11}$$

Now we find, with proper choice of α and N, the lower bound for the $\dot{H}^s \times \dot{H}^s$ norm of F_3.

If $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$ and $\xi - \xi_1 \in I_1$ then $|\xi_1| \sim N$, $|\xi - \xi_1| \sim N$ and $|\xi| \sim \alpha$. Therefore,

$$|\xi\xi_1(\xi-\xi_1)| \sim N^2\alpha.$$

For $0 < \epsilon \ll 1$, choose N and α such that $N^2\alpha = N^{-\epsilon}$. Hence for $\xi_1 \in A_3(\xi)$ we have, for fixed t and large N,

$$\Big|\frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\Big| \geq C > 0. \tag{2.5.12}$$

Now,

$$\|F_3\|^2_{\dot{H}^s \times \dot{H}^s} \sim \int_{\alpha}^{3\alpha} |\xi|^{2s}\alpha^{-2}N^{-4s}|\xi|^2(a'^2+b'^2)\Big| \int_{A_3(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\, d\xi_1\Big|^2 d\xi.$$

Using the Mean Value Theorem for integrals and (2.5.12) it is easy to see that,

$$\Big|\int_{A_3(\xi)}\frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\,d\xi_1\Big|\geq C|A_3(\xi)|. \tag{2.5.13}$$

Also, it is easy to see that $|A_3(\xi)|\sim\alpha$. In fact, $|A_3(\xi)|\leq|I_1|+|I_2|\leq 2\alpha$. On the other hand, if $\xi\in(7\alpha/4,\,9\alpha/4)\subset(\alpha,3\alpha)$ then $[-N+\alpha/4,\,-N+3\alpha/4]\subset A_3(\xi)$ and we get $|A_3(\xi)|\geq\alpha/2$.

Now using (2.5.13), we get,

$$\|F_3\|^2_{\dot H^s\times\dot H^s}\geq CN^{-4s}\alpha^{-2}(a'^2+b'^2)\int_{\frac{7}{4}\alpha}^{\frac{9}{4}\alpha}|\xi|^{2s+2}\alpha^2\,d\xi\sim CN^{-4s}\alpha^{2s+3}.$$

Therefore,

$$\|F_3\|^2_{\dot H^s\times\dot H^s}\geq CN^{-2s}\alpha^{s+\frac{3}{2}}. \tag{2.5.14}$$

Observe that supp $\widehat{\mathrm{F_1}}\subseteq[-2\mathrm{N},-2\mathrm{N}+2\alpha]$, supp $\widehat{\mathrm{F_2}}\subseteq[2\mathrm{N}+2\alpha,2\mathrm{N}+4\alpha]$ and supp $\widehat{\mathrm{F_3}}\subseteq[\alpha,3\alpha]$, which are clearly disjoint, therefore using (2.5.8), (2.5.9), (2.5.10), (2.5.11) and (2.5.14) in (2.5.6) we obtain,

$$\begin{aligned}1\sim\|\Phi\|^2_{\dot H^s\times\dot H^s}&\geq\|W_2(\cdot,t)\|^2_{\dot H^s\times\dot H^s}\\&\geq\|F_3(\cdot,t)\|^2_{\dot H^s\times\dot H^s}\\&\geq CN^{-2s}\alpha^{s+\frac{3}{2}}\\&=CN^{-4s-3}N^{-\epsilon(s+\frac{3}{2})}.\end{aligned}$$

Hence

$$N^{-4s-3}\leq CN^{\epsilon(s+3/2)}. \tag{2.5.15}$$

Case I: If $-3/2<s<-3/4$ then $s+3/2>0$ and (2.5.15) gives $N^{(-4s-3)-\epsilon(s+3/2)}\leq C$ which is a contradiction for $N\gg 1$ if we choose $0<\epsilon<(-4s-3)/(s+3/2)$.

Case II: If $s \leq -3/2$ then $s + 3/2 \leq 0$ and (2.5.15) gives $N^{-4s-3} \leq C$, which is again a contradiction for $N \gg 1$.

Hence for $s < -3/4$, (2.5.6) fails to hold for our choice of Φ, which completes the proof of the theorem. □

2.6 Comments

The method used in Chapter 2 can also be applied to obtain analogous results to the IVP associated to the following coupled KdV system of Nutku and Oğuz [69]

$$\begin{cases} u_t = u_{xxx} + 2\alpha u u_x + v v_x + (uv)_x \\ v_t = v_{xxx} + 2\beta v v_x + u u_x + (uv)_x, \end{cases} \tag{2.6.16}$$

where α, β are constants.

There is still ample room to improve and extend the results obtained in this chapter. Also there is strong possibility to use these techniques to new models as well. In what follows we mention some open problem that may be of interest in the future.

It would be interesting to obtain the best possible global well-posedness result for the IVP associated to (2.1.1), i.e., for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$. For this, one can expect to extend the variant of the method of Bourgain [13] introduced by Colliander et. al. [24] in the KdV context. Also, it would be interesting to get global solution without imposing restriction on the coefficients. The presence of the arbitrary constants suggests one to proceed in some other way to implement the theory used in this work. Another nice problem is to utilize the recent work of Lopes [61] to study the orbital stability of the solitary wave solutions to the system (2.1.1).

The study of the Cauchy problem associated to the following new coupled KdV and mKdV systems [18], [34], [92],

$$\begin{cases} u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 6vv_x \\ v_t = -v_{xxx} + 3uv_x \end{cases} \tag{2.6.17}$$

and

$$\begin{cases} p_t = \frac{1}{2}p_{xxx} - 3p^2p_x + 3(qq_x)_x + 3(pq^2)_x \\ q_t = -q_{xxx} - 3(qp_x)_x + 6pqp_x + 3(p^2 - q^2)q_x. \end{cases} \tag{2.6.18}$$

will also be of interest.

We believe that the methods employed to the model considered here can be utilized (with necessary modification and generalization) to obtain similar results to these new models but it has to be done.

Chapter 3

Coupled System of the mKdV Equations

3.1 Introduction

This chapter is concerned about the global solution to the IVP (1.3.63) associated to the system of the modified Korteweg-de Vries (mKdV) equations.

The system (1.3.63) has a structure of the mKdV equation coupled through nonlinear effects and is a special case of a broad class of nonlinear evolution equations of physical interest (see for eg [1]). Many complex physical phenomena can be modeled as mKdV equation. In recent years much effort has been made to study the mKdV model (see for example [10], [12], [27], [52], [80] and references therein). This model has also been studied using inverse scattering theory (see [64], [76] and references therein).

The Cauchy problem as well as the existence and stability of solitary wave solutions to (1.3.63) is widely studied in the literature (see for example [67] and [2]). Recently using the argument developed by Kenig, Ponce and Vega [52] in context of the mKdV equation, Montenegro [67] proved that the IVP (1.3.63) is locally well-posed for given data $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1/4$. More precisely, the following theorem has been proved in [67].

Theorem 3.1. *Let $s \geq 1/4$. Then for all $(\phi, \psi) \in X^s$, there exist $T = T(\|\phi\|_{H^{1/4}}, \|\psi\|_{H^{1/4}})$ [in fact $T \sim c\|(\phi, \psi)\|_{X^{1/4}}^{-4} > 0$] and a unique solution $(u(t), v(t))$ to the IVP (1.3.63) such that*

$$(u, v) \in C([-T, T] : X^s)$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} < \infty, \qquad \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} < \infty, \tag{3.1.1}$$

$$\|\partial_x u\|_{L_x^{20} L_T^{5/2}} < \infty, \qquad \|\partial_x v\|_{L_x^{20} L_T^{5/2}} < \infty, \tag{3.1.2}$$

$$\|D_x^s u\|_{L_x^5 L_T^{10}} < \infty, \qquad \|D_x^s v\|_{L_x^5 L_T^{10}} < \infty, \tag{3.1.3}$$

$$\|u\|_{L_x^4 L_T^\infty} < \infty, \qquad \|v\|_{L_x^4 L_T^\infty} < \infty. \tag{3.1.4}$$

Moreover, for any $T' \in (0,T)$, there exists a neighborhood $\mathcal{V}$ of (ϕ,ψ) in X^s such that the map $(\tilde{\phi},\tilde{\psi}) \mapsto (\tilde{u},\tilde{v})$ from $\mathcal{V}$ into the class defined by (3.1.1) to (3.1.4) with T' in place of T is Lipschitz.

Using the conservation laws (1.3.61) and (1.3.62) satisfied by the flow of (1.3.63), the local solution can be extended to the global one for the initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$. Hence, there is a gap in Sobolev indices between the existence of the local and global solution to the IVP (1.3.63). In this work, we will fill this gap to some extent.

3.2 Main Result and the Scheme of the Proof

Here we further refine the high-low frequency technique introduced by Bourgain [13] and more simplified by Fonseca, Linares and Ponce [27], in the context of the mKdV equation. For this we exploit the uniform bound of the solution (see (3.2.1) below) obtained by using iteration in the energy space. With proper choice of the Sobolev indices we develop an iteration process below the energy space and prove that the IVP (1.3.63) is globally well-posed for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 4/9$. More precisely, we prove the following result.

Theorem 3.2. *For any $(\phi,\psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 4/9$, the unique solution to the IVP (1.3.63) provided by Theorem 3.1 extends to any time interval $[0,T]$.*

To prove this theorem we use the sharp smoothing effects present in the solution of the linear problem associated to the IVP (1.3.63) combined with the iteration process introduced by Bourgain [13]. The proof will be carried out in two steps. In the first step we closely follow the modified techniques developed by Fonseca, Linares and Ponce [27] to perform iteration in the energy space and prove that the local solution to the IVP (1.3.63) can be extended to a global one for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 3/5$. Moreover, we obtain that the solution grows according as

$$\sup_{[0,T]} \|(u(t), v(t))\|_{H^s \times H^s} \leq cT^{2s(1-s)/(5s-3)}, \qquad 3/5 < s < 1, \tag{3.2.1}$$

with $N = N(T) \sim T^{2/(5s-3)}$, sufficiently large.

In the second step, we take $s_0 \in (3/5, 1)$ and the data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $1/4 \leq s < s_0$. Utilizing the uniform bound (3.2.1) of the solution in $H^{s_0}(\mathbb{R}) \times H^{s_0}(\mathbb{R})$ obtained in the first step, we develop an iteration process in this space by controlling the involved norms and complete the proof (for details, see proof of the Theorem 3.2 below).

3.3 Linear Estimates

In this section we give some linear estimates associated to the IVP (1.3.63). These estimates are not new and can be found in literature. We will not give the details of the proofs rather we just sketch the idea of the proof and mention the references where these can be found. Let $U(t)$ be the group generated by the operator ∂_x^3. First let us state the smoothing effects.

Theorem 3.3. *If $\phi \in L^2(\mathbb{R})$, then*

$$\|\partial_x U(t)\phi\|_{L_x^\infty L_t^2} \leq \|\phi\|_{L^2}. \tag{3.3.1}$$

If $g \in L_x^1 L_t^2$ then for any $T > 0$

$$\|\partial_x \int_0^t U(t-t')g(\cdot, t')\, dt'\|_{L_T^\infty L_x^2} \leq \|g\|_{L_x^1 L_T^2}. \tag{3.3.2}$$

If $g \in L_x^{2/(1+\theta)} L_t^2$, $0 \leq \theta \leq 1$, then for any $T > 0$

$$\|D_x^\theta \int_0^t U(t-t')g(\cdot, t')\, dt'\|_{L_T^\infty L_x^2} \leq cT^{\frac{1}{2}(1-\theta)} \|g\|_{L_x^{2/(1+\theta)} L_T^2}. \tag{3.3.3}$$

Proof. For the proof of the homogeneous smoothing effect (3.3.1) see section 4 in [55] (see also [52]). Inequality (3.3.2) is the dual version of the smoothing effect (3.3.1). The estimate (3.3.3) can be found in [20]. In fact, the Minkowski integral inequality and the Cauchy-Schwarz inequality yield,

$$\|\partial_x \int_0^t U(t-t')g(\cdot, t')\, dt'\|_{L_x^2} \leq \int_0^T \|D_x g(\cdot, t')\|_{L_x^2}\, dt' \leq T^{1/2} \|D_x g\|_{L_x^2 L_T^2}. \tag{3.3.4}$$

Now, application of Stein's interpolation (see [81]) between the dual version of the smoothing effect (3.3.2) and the estimate (3.3.4) by considering the analytic family of operators $T_z f = D_x^{-z}(\int_0^t \partial_x U(t-t')f(\cdot,t')\,dt')$, $z \in \mathbb{C}$, $0 \leq \Re z \leq 1$ gives,

$$\|\partial_x \int_0^t U(t-t')g(\cdot,t')\,dt'\|_{L_T^\infty L_x^2} \leq cT^{\theta/2}\|D_x^\theta g\|_{L_x^{2/(2-\theta)}L_T^2}, \tag{3.3.5}$$

which implies the required estimate. □

Observe that, if $D_x^\theta \phi \in L^2$ for $0 < \theta \leq 1$, then using Stein's interpolation between the homogeneous smoothing effect (3.3.1) and $\|U(t)f\|_{L_x^2 L_T^2} \leq T^{1/2}\|f\|_{L_x^2}$ we get,

$$\|\partial_x U(t)\phi\|_{L_x^{2/\theta}L_T^2} \leq cT^{\theta/2}\|D_x^\theta \phi\|_{L^2}. \tag{3.3.6}$$

In what follows, we record the maximal function estimates and other mmixed norm estimates that are used in the proofs of the main results.

Theorem 3.4. *If $\phi \in H^{1/4}$, then*

$$\|U(t)\phi\|_{L_x^4 L_T^\infty} \leq c\|D_x^{1/4}\phi\|_{L^2}. \tag{3.3.7}$$

If $\phi \in H^s,\ s > 3/4$ and $0 < T < 1$ then

$$\|U(t)\phi\|_{L_x^2 L_T^\infty} \leq c\|\phi\|_{H^s}. \tag{3.3.8}$$

If $\phi \in H^{(1+2\theta)/4}$, $0 \leq \theta < 1$ and $0 < T < 1$ then

$$\|U(t)\phi\|_{L_x^{4/(1+\theta)}L_T^\infty} \leq c\|\phi\|_{H^{(1+2\theta)/4}}. \tag{3.3.9}$$

Proof. The proof of the estimates (3.3.7) and (3.3.8) can be found in [54] and [58]. The estimate (3.3.9) can be obtained by interpolating (3.3.7) and (3.3.8). □

Theorem 3.5. *If $\phi \in L^2(\mathbb{R})$, then*

$$\|U(t)\phi\|_{L_x^5 L_t^{10}} \leq c\|\phi\|_{L^2} \tag{3.3.10}$$

and

$$\|\partial_x U(t)\phi\|_{L_x^{20}L_t^{5/2}} \leq c\|D_x^{1/4}\phi\|_{L^2}. \tag{3.3.11}$$

If $g \in L_x^{5/4} L_t^{10/9}$

$$\| \int_0^t U(t-t')g(\cdot,t')\,dt' \|_{L_x^5 L_t^{10}} \leq c\|g\|_{L_x^{5/4} L_t^{10/9}}. \tag{3.3.12}$$

Proof. The estimates of this theorem can be found in [52]. The estimate (3.3.10) follows by interpolating (3.3.1) and (3.3.8). The estimate (3.3.11) follows by using Stein's interpolation between (3.3.1) and (3.3.7). The estimate (3.3.12) follows by using interpolation in BMO spaces, see [52]. □

Theorem 3.6. *Let* $1/4 \leq \theta \leq 1$. *If* $D_x^\theta \phi \in L^2$, *then*

$$\|D_x U(t)\phi\|_{L_x^{40/(20\theta-3)} L_t^{5/2}} \leq cT^{\theta/2-1/8}\|D_x^\theta \phi\|_{L_x^2}. \tag{3.3.13}$$

Proof. The proof of this estimate can be found in [20] which follows by interpolating (3.3.6) and

$$\|D_x U(t)\phi\|_{L_x^{5/\theta} L_t^{10/(5-4\theta)}} \leq \|D_x^\theta \phi\|_{L_x^2},\ 0 \leq \theta \leq 1. \tag{3.3.14}$$

The estimate (3.3.14) can be obtained by interpolating the homogeneous smoothing effect (3.3.1) and the maximal function estimate (3.3.7). □

Finally we have the Leibniz's rule for fractional derivatives whose proof is given in [52].

Theorem 3.7. *Let* $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$, $\alpha_1 + \alpha_2 = \alpha$. *Let* $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ *be such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \leq c\|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}} \tag{3.3.15}$$

Moreover, for $\alpha_1 = 0$ *the value* $q_1 = \infty$ *is allowed.*

3.4 Preliminary Results

We decompose the given data $(\phi,\psi) \in X^s,\ s < 1$ to low and high frequency parts as,

$$\begin{cases} \phi(x) = (\chi_{\{|\xi|\leq N\}}\hat{\phi}(\xi))^\vee(x) + (\chi_{\{|\xi|>N\}}\hat{\phi}(\xi))^\vee(x) := \phi_1(x) + \phi_2(x) \\ \psi(x) = (\chi_{\{|\xi|\leq N\}}\hat{\psi}(\xi))^\vee(x) + (\chi_{\{|\xi|>N\}}\hat{\psi}(\xi))^\vee(x) := \psi_1(x) + \psi_2(x), \end{cases} \tag{3.4.1}$$

where $N \gg 1$ arbitrary but fixed.

Then we have, $(\phi_1, \psi_1) \in X^{\beta},\ 0 \le \beta \le 1$ and $(\phi_2, \psi_2) \in X^{\rho}, \quad 0 \le \rho \le s < 1.$

As discussed in the introduction, we evolve (ϕ_1, ψ_1) according to the IVP*

$$\begin{cases} u_{1t} + u_{1xxx} + (u_1 v_1^2)_x = 0 \\ v_{1t} + v_{2xxx} + (u_1^2 v_1)_x = 0 \\ u_1(x,0) = \phi_1(x), \quad v_1(x,0) = \psi_1(x), \end{cases} \tag{3.4.2}$$

which is the same as the IVP (1.3.63). We evolve (ϕ_2, ψ_2) according to the difference equation

$$\begin{cases} u_{2t} + u_{2xxx} + ((u_1+u_2)(v_1+v_2)^2)_x - (u_1 v_1^2)_x = 0 \\ v_{2t} + v_{2xxx} + ((u_1+u_2)^2(v_1+v_2))_x - (u_1^2 v_1)_x = 0 \\ u_2(x,0) = \phi_2(x), \quad v_2(x,0) = \psi_2(x), \end{cases} \tag{3.4.3}$$

with coefficients depending on the solution (u_1, v_1) to the IVP (3.4.2). It is clear that $u = u_1 + u_2$ and $v = v_1 + v_2$ solve the IVP (1.3.63). For simplicity, let us write (3.4.3) as

$$\begin{cases} u_{2t} + u_{2xxx} + F = 0 \\ v_{2t} + v_{2xxx} + G = 0 \\ u_2(x,0) = \phi_2(x), \quad v_2(x,0) = \psi_2(x), \end{cases} \tag{3.4.4}$$

where

$$\begin{aligned} F =& 2u_1v_1v_{2x} + 2u_1v_2v_{1x} + 2v_1v_2u_{1x} + 2u_1v_2v_{2x} + 2u_2v_1v_{1x} + 2u_2v_1v_{2x} \\ &+ 2u_2v_2v_{1x} + 2v_1v_2u_{2x} + 2u_2v_2v_{2x} + v_2^2u_{1x} + v_1^2u_{2x} + v_2^2u_{2x} \end{aligned} \tag{3.4.5}$$

and

$$\begin{aligned} G =& 2v_1u_1u_{2x} + 2v_1u_2u_{1x} + 2u_1u_2v_{1x} + 2v_1u_2u_{2x} + 2v_2u_1u_{1x} + 2v_2u_1u_{2x} \\ &+ 2v_2u_2u_{1x} + 2u_1u_2v_{2x} + 2v_2u_2u_{2x} + u_2^2v_{1x} + u_1^2v_{2x} + u_2^2v_{2x}. \end{aligned} \tag{3.4.6}$$

Note that from Theorem 3.1 we have the existence result for the IVP (3.4.2). To get the existence result for the IVP (3.4.4) we need the following theorem.

*We use the notations $u_{1t} := (u_1)_t$, $u_{1x} := (u_1)_x$ and similar for other terms.

Theorem 3.8. *Suppose the initial data (ϕ_1, ψ_1) of the IVP (3.4.2) satisfy*

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \le c \\ \|(\phi_1, \psi_1)\|_{X^1} \le cN^{1-s}. \end{cases} \tag{3.4.7}$$

Then for the existence time $T \sim c\|(\phi_1, \psi_1)\|_{X^{1/4}}^{-4} \sim cN^{-(1-s)}$ obtained in Theorem 3.1
(i) The solution (u_1, v_1) to the IVP (3.4.2) satisfies,

$$\sup_t \|(u_1(t), v_1(t))\|_{X^1} = \sup_t [\|u_1(t)\|_{H^1} + \|v_1(t)\|_{H^1}] \le cN^{1-s}. \tag{3.4.8}$$

(ii) Moreover, for any $\beta \in [1/4, 1)$, the solution (u_1, v_1) to the IVP (3.4.2) satisfies,

$$|\!|\!|(u_1, v_1)|\!|\!|_\beta \sim N^{(1-s)\beta}, \tag{3.4.9}$$

where $|\!|\!|(u_1, v_1)|\!|\!|_\beta = \max\{|\!|\!|u_1|\!|\!|_\beta, |\!|\!|v_1|\!|\!|_\beta\}$ and,

$$\begin{aligned} |\!|\!|f|\!|\!|_\beta = \|f\|_{L^\infty_T H^\beta} + \|D_x^\beta \partial_x f\|_{L^\infty_x L^2_T} + \|\partial_x f\|_{L^{20}_x L^{5/2}_T} \\ + \|D_x^\beta f\|_{L^5_x L^{10}_T} + \|f\|_{L^4_x L^\infty_T} + \|\partial_x f\|_{L^\infty_x L^2_T}. \end{aligned} \tag{3.4.10}$$

Proof. The proof of (3.4.8) follows by using the conservation laws (1.3.61) and (1.3.62) combined with the Gagliardo-Nirenberg inequality. The estimate (3.4.9) can be obtained by using the hypothesis (3.4.7) and the local well-posedness result. □

The following theorem provides the existence result for the IVP (3.4.4).

Theorem 3.9. *Let $(\phi_2, \psi_2) \in X^s$, $s \ge 1/4$ and (u_1, v_1) be the unique solution given by Theorem 3.8. Then there exists a unique solution (u_2, v_2) to the IVP (3.4.4) in the same interval of existence of (u_1, v_1), $[0, T]$ such that,*

$$(u_2, v_2) \in C([0, T] : X^s)$$

$$\|D_x^s \partial_x u_2\|_{L^\infty_x L^2_T} < \infty, \qquad \|D_x^s \partial_x v_2\|_{L^\infty_x L^2_T} < \infty, \tag{3.4.11}$$

$$\|\partial_x u_2\|_{L^{20}_x L^{5/2}_T} < \infty, \qquad \|\partial_x v_2\|_{L^{20}_x L^{5/2}_T} < \infty, \tag{3.4.12}$$

$$\|D_x^s u_2\|_{L^5_x L^{10}_T} < \infty, \qquad \|D_x^s v_2\|_{L^5_x L^{10}_T} < \infty, \tag{3.4.13}$$

$$\|u_2\|_{L^4_x L^\infty_T} < \infty, \qquad \|v_2\|_{L^4_x L^\infty_T} < \infty. \tag{3.4.14}$$

Proof. We will prove this theorem following the argument in [52]. As in [52] we will give details only for the case $s = 1/4$, for this we consider the equivalent integral equation associated to the IVP (3.4.4), i.e,

$$\begin{cases} u_2(t) = U(t)\phi_2 - \int_0^t U(t-t')F(t')\,dt' \\ v_2(t) = U(t)\psi_2 - \int_0^t U(t-t')G(t')\,dt', \end{cases} \tag{3.4.15}$$

where F and G are defined in (3.4.5) and (3.4.6) respectively.

For some $a > 0$, let us define a ball

$$\mathfrak{X}_{a,T} = \{(u_2, v_2) \in C([0,T] : X^{1/4}(\mathbb{R})) : \|\!|(u_2, v_2)|\!\|_{1/4} < a\},$$

where

$$\|\!|(u_2, v_2)|\!\|_{1/4} = \max\{\|\!|u_2|\!\|_{1/4}, \|\!|v_2|\!\|_{1/4}\},$$

with

$$\begin{aligned} \|\!|f|\!\|_{1/4} = \|f\|_{L_T^\infty H^{1/4}} + \|D_x^{1/4}\partial_x f\|_{L_x^\infty L_T^2} + \|\partial_x f\|_{L_x^{20} L_T^{5/2}} \\ + \|D_x^{1/4} f\|_{L_x^5 L_T^{10}} + \|f\|_{L_x^4 L_T^\infty} + \|\partial_x f\|_{L_x^\infty L_T^2}. \end{aligned} \tag{3.4.16}$$

Now, we define the following applications

$$\begin{cases} \Phi_{\phi_2}[u_2, v_2] = U(t)\phi_2 - \int_0^t U(t-t')F(t')\,dt' \\ \Psi_{\psi_2}[u_2, v_2] = U(t)\psi_2 - \int_0^t U(t-t')Gt')\,dt', \end{cases} \tag{3.4.17}$$

and show that there exist $a > 0$ and $T > 0$ such that $\Phi \times \Psi$ maps $\mathfrak{X}_{a,T}$ into $\mathfrak{X}_{a,T}$ and is a contraction.

Using the linear estimates established in section 3.3 we obtain,

$$\begin{aligned} \|\!|(\Phi, \Psi)|\!\|_{1/4} \leq c\|(\phi_2, \psi_2)\|_{X^{1/4}} + cT^{1/2}\{\|D_x^{1/4} F\|_{L_x^2 L_T^2} \\ + \|F\|_{L_x^2 L_T^2} + \|D_x^{1/4} G\|_{L_x^2 L_T^2} + \|G\|_{L_x^2 L_T^2}\} \\ = c\|(\phi_2, \psi_2)\|_{X^{1/4}} + cT^{1/2}\{A_1 + A_2 + A_3 + A_4\}. \end{aligned} \tag{3.4.18}$$

Now using the definition of F we get,

$$
\begin{aligned}
A_1 \leq\ & c\big[\|D_x^{1/4}(u_1v_1v_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_1v_2v_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_1v_2u_{1x})\|_{L_x^2L_T^2} \\
& + \|D_x^{1/4}(u_1v_2v_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_2v_1v_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_2v_1v_{2x})\|_{L_x^2L_T^2} \\
& + \|D_x^{1/4}(u_2v_2v_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_1v_2u_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_2v_2v_{2x})\|_{L_x^2L_T^2} \\
& + \|D_x^{1/4}(v_2^2u_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_1^2u_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_2^2u_{2x})\|_{L_x^2L_T^2}\big] \\
:=\ & A_{1,1} + A_{1,2} + \cdots + A_{1,12}.
\end{aligned}
\tag{3.4.19}
$$

Also, we can have the similar expressions for A_2, A_3 and A_4.

Using the Leibniz's rule for fractional derivative and Hölder's inequality we get,

$$
\begin{aligned}
A_{1,1} &= c\|D_x^{1/4}(u_1v_1v_{2x})\|_{L_x^2L_T^2} \\
&\leq c\big[\|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}}\|v_{2x}\|_{L_x^{20}L_T^{5/2}} + \|u_1v_1D_x^{1/4}v_{2x}\|_{L_x^2L_T^2} \\
&\quad + \|v_{2x}D_x^{1/4}(u_1v_1)\|_{L_x^2L_T^2}\big] \\
&\leq c\big[\|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}}\|v_{2x}\|_{L_x^{20}L_T^{5/2}} + \|u_1\|_{L_x^4L_T^\infty}\|v_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}v_{2x}\|_{L_x^\infty L_T^2}\big] \\
&\leq c\big[\|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}}\|\!|v_2|\!\|_{1/4} + \|\!|u_1|\!\|_{1/4}\|\!|v_1|\!\|_{1/4}\|\!|v_2|\!\|_{1/4}\big].
\end{aligned}
\tag{3.4.20}
$$

Again, using the Leibniz's rule for fractional derivative and Hölder's inequality we have,

$$
\begin{aligned}
\|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}} \leq\ & c\big[\|u_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}v_1\|_{L_x^5L_T^{10}} + \|u_1D_x^{1/4}v_1\|_{L_x^{20/9}L_T^{10}} \\
& + \|v_1D_x^{1/4}u_1\|_{L_x^{20/9}L_T^{10}}\big] \\
\leq\ & c\big[\|u_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}v_1\|_{L_x^5L_T^{10}} + \|v_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}u_1\|_{L_x^5L_T^{10}}\big] \\
\leq\ & c\|\!|u_1|\!\|_{1/4}\|\!|v_1|\!\|_{1/4}.
\end{aligned}
\tag{3.4.21}
$$

Inserting (3.4.21) in (3.4.20) we obtain,

$$
A_{1,1} \leq c\|\!|(u_1,v_1)|\!\|_{1/4}^2\|\!|(u_2,v_2)|\!\|_{1/4}.
$$

Also, we can get the analogous estimates for $A_{1,j},\ j = 2, 3, \ldots, 12$.

Therefore, from (3.4.19) we obtain,

$$A_1 \le c\big[\|\!|(u_1,v_1)\|\!|_{1/4}^2 \|\!|(u_2,v_2)\|\!|_{1/4} + \|\!|(u_1,v_1)\|\!|_{1/4}\|\!|(u_2,v_2)\|\!|_{1/4}^2 + \|\!|(u_2,v_2)\|\!|_{1/4}^3\big].$$

Using the similar argument we can get for $j = 2, 3, 4$,

$$A_j \le c\big[\|\!|(u_1,v_1)\|\!|_{1/4}^2 \|\!|(u_2,v_2)\|\!|_{1/4} + \|\!|(u_1,v_1)\|\!|_{1/4}\|\!|(u_2,v_2)\|\!|_{1/4}^2 + \|\!|(u_2,v_2)\|\!|_{1/4}^3\big].$$

Hence,

$$\begin{aligned}\|\!|(\Phi,\Psi)\|\!|_{1/4} \le c\|(\phi_2,\psi_2)\|_{H^{1/4}} + cT^{1/2}\big\{&\|\!|(u_1,v_1)\|\!|_{1/4}^2 \\ &+ \|\!|(u_1,v_1)\|\!|_{1/4}\|\!|(u_2,v_2)\|\!|_{1/4} + \|\!|(u_2,v_2)\|\!|_{1/4}^2\big\}\|\!|(u_2,v_2)\|\!|_{1/4}.\end{aligned} \tag{3.4.22}$$

Let us set $a = 2c\max\{\|(\phi_1,\psi_1)\|_{X^{1/4}}, \|(\phi_2,\psi_2)\|_{X^{1/4}}\}$. With this choice, if we take T such that $ca^2T^{1/2} < 1/10$, then (3.4.22) yields,

$$\|\!|(\Phi,\Psi)\|\!|_{1/4} \le \frac{a}{2} + \frac{3}{10}\,a < a.$$

Therefore, $\Phi\times\Psi$ maps $\mathcal{X}_{a,T}$ into $\mathcal{X}_{a,T}$. Using the same argument we can show that $\Phi\times\Psi$ is a contraction. The rest of the proof follows a standard argument. □

In what follows we need the following results.

Corollary 3.1. *Let (u_1,v_1) and (u_2,v_2) be solutions to the IVPs (3.4.2) and (3.4.4) with initial data $(\phi_1,\psi_1),(\phi_2,\psi_2)\in X^s$, $s\ge 1/4$ respectively. For $1/4<\rho\le s<1$, let (ϕ_2,ψ_2) satisfies,*

$$\|(\phi_2,\psi_2)\|_{X^\rho} \sim N^{\rho-s} \tag{3.4.23}$$

and (ϕ_1,ψ_1) satisfies the conditions of Theorem 3.8.

If

$$\|\!|\!|(u_2,v_2)\|\!|\!|_\rho = \max\{\|\!|\!|u_2\|\!|\!|_\rho, \|\!|\!|v_2\|\!|\!|_\rho\},$$

where,

$$\|\!|\!|f\|\!|\!|_\rho = \|f\|_{L^\infty_T H^\rho} + \|D_x^\rho\partial_x f\|_{L^\infty_x L^2_T} + \|\partial_x f\|_{L^{20}_x L^{5/2}_T} + \|D_x^\rho f\|_{L^5_x L^{10}_T}.$$

Then,

$$|||(u_2, v_2)|||_\rho \sim N^{\rho - s}.$$

Proof. Using the definition of $|||\cdot|||_\rho$ and the linear estimates established in section 3.3 we obtain,

$$\begin{aligned}|||(u_2, v_2)|||_\rho &\le c\|(\phi_2, \psi_2)\|_{X^\rho} + cT^{1/2}\big\{\|D_x^\rho F\|_{L_x^2 L_T^2} + \|F\|_{L_x^2 L_T^2} + \|D_x^\rho G\|_{L_x^2 L_T^2} + \|G\|_{L_x^2 L_T^2}\big\}\\ &= c\|(\phi_2, \psi_2)\|_{X^\rho} + B_1 + B_2 + B_3 + B_4.\end{aligned} \tag{3.4.24}$$

From the definition of F we get,

$$\begin{aligned}B_1 \le cT^{1/2}\big[&\|D_x^\rho(u_1 v_1 v_{2x})\|_{L_x^2 L_T^2} + \|D_x^\rho(u_1 v_2 v_{1x})\|_{L_x^2 L_T^2} + \|D_x^\rho(v_1 v_2 u_{1x})\|_{L_x^2 L_T^2}\\ &+ \|D_x^\rho(u_1 v_2 v_{2x})\|_{L_x^2 L_T^2} + \|D_x^\rho(u_2 v_1 v_{1x})\|_{L_x^2 L_T^2} + \|D_x^\rho(u_2 v_1 v_{2x})\|_{L_x^2 L_T^2}\\ &+ \|D_x^\rho(u_2 v_2 v_{1x})\|_{L_x^2 L_T^2} + \|D_x^\rho(v_1 v_2 u_{2x})\|_{L_x^2 L_T^2} + \|D_x^\rho(u_2 v_2 v_{2x})\|_{L_x^2 L_T^2}\\ &+ \|D_x^\rho(v_2^2 u_{1x})\|_{L_x^2 L_T^2} + \|D_x^\rho(v_1^2 u_{2x})\|_{L_x^2 L_T^2} + \|D_x^\rho(v_2^2 u_{2x})\|_{L_x^2 L_T^2}\big]\\ := B_{1,1} + B_{1,2} + &\cdots + B_{1,12}.\end{aligned} \tag{3.4.25}$$

Using the Leibniz's rule for fractional derivative and Hölder's inequality one gets,

$$\begin{aligned}B_{1,1} =\; & T^{1/2}\|D_x^\rho(u_1 v_1 v_{2x})\|_{L_x^2 L_T^2}\\ \le\; & cT^{1/2}\big[\|D_x^\rho(u_1 v_1)\|_{L_x^{20/9} L_T^{10}}\|v_{2x}\|_{L_x^{20} L_T^{5/2}} + \|v_{2x} D_x^\rho(u_1 v_1)\|_{L_x^2 L_T^2}\\ & + \|u_1 v_1 D_x^\rho v_{2x}\|_{L_x^2 L_T^2}\big]\\ \le\; & cT^{1/2}\big[\|D_x^\rho(u_1 v_1)\|_{L_x^{20/9} L_T^{10}}\|v_{2x}\|_{L_x^{20} L_T^{5/2}} + \|u_1\|_{L_x^4 L_T^\infty}\|v_1\|_{L_x^4 L_T^\infty}\|D_x^\rho v_{2x}\|_{L_x^\infty L_T^2}\big]\\ \le\; & cT^{1/2}\big[|||v_2|||_{1/4}\|D_x^\rho(u_1 v_1)\|_{L_x^{20/9} L_T^{10}} + |||u_1|||_{1/4}|||v_1|||_{1/4}|||v_2|||_\rho\big].\end{aligned} \tag{3.4.26}$$

Again using the Leibniz's rule for fractional derivative we get,

$$\begin{aligned}\|D_x^\rho(u_1 v_1)\|_{L_x^{20/9} L_T^{10}} &\le c\big[\|u_1\|_{L_x^4 L_T^\infty}\|D_x^\rho v_1\|_{L_x^5 L_T^{10}} + \|u_1 D_x^\rho v_1\|_{L_x^{20/9} L_T^{10}} + \|v_1 D_x^\rho u_1\|_{L_x^{20/9} L_T^{10}}\big]\\ &\le c\big[|||u_1|||_{1/4}|||v_1|||_\rho + \|v_1\|_{L_x^4 L_T^\infty}\|D_x^\rho u_1\|_{L_x^5 L_T^{10}}\big]\\ &\le c\big[|||u_1|||_{1/4}|||v_1|||_\rho + c|||v_1|||_{1/4}|||u_1|||_\rho\big].\end{aligned} \tag{3.4.27}$$

Inserting (3.4.27) in (3.4.26) we obtain,

$$B_{1,1} \leq cT^{1/2}[\|(u_1,v_1)\|_{1/4}\|(u_1,v_1)\|_{\rho}\|(u_2,v_2)\|_{1/4} + \|(u_1,v_1)\|_{1/4}^2 \|\!|(u_2,v_2)|\!\|_{\rho}]. \tag{3.4.28}$$

Using the similar argument it is easy to get, for $j = 2, 3, 5, 11$,

$$B_{1,j} \leq cT^{1/2}\big[\|(u_1,v_1)\|_{1/4}\|(u_1,v_1)\|_{\rho}\|(u_2,v_2)\|_{1/4} + \|(u_1,v_1)\|_{1/4}^2 \|\!|(u_2,v_2)|\!\|_{\rho}\big], \tag{3.4.29}$$

for $j = 4, 6, 7, 8, 10$,

$$B_{1,j} \leq cT^{1/2}\big[\|(u_1,v_1)\|_{\rho}\|(u_2,v_2)\|_{1/4}^2 + \|(u_1,v_1)\|_{1/4}\|(u_2,v_2)\|_{1/4}\|\!|(u_2,v_2)|\!\|_{\rho}\big], \tag{3.4.30}$$

and for $j = 9, 12$,

$$B_{1,j} \leq cT^{1/2}\big[\|(u_2,v_2)\|_{1/4}^2 \|\!|(u_2,v_2)|\!\|_{\rho}\big]. \tag{3.4.31}$$

Now, using Theorem 3.8, Theorem 3.9 and $T \sim N^{-(1-s)}$ we get from (3.4.28) - (3.4.31),

$$B_{1,j} \leq cN^{\rho-s} + \frac{1}{20}\|\!|(u_2,v_2)|\!\|_{\rho}, \qquad j = 1, 2, 3, 5, 11, \tag{3.4.32}$$

$$B_{1,j} \leq cN^{\rho-s} + cN^{-\frac{3}{4}s}\|\!|(u_2,v_2)|\!\|_{\rho}, \quad j = 4, 6, 7, 8, 10 \tag{3.4.33}$$

and

$$B_{1,j} \leq cN^{-\frac{3}{2}s}\|\!|(u_2,v_2)|\!\|_{\rho}, \qquad j = 9, 12. \tag{3.4.34}$$

The use of (3.4.32) - (3.4.34) in (3.4.25) yields,

$$B_1 \leq c\frac{1}{4}\|\!|(u_2,v_2)|\!\|_{\rho} + cN^{-\frac{3}{4}s}\|\!|(u_2,v_2)|\!\|_{\rho} + cN^{\rho-s}. \tag{3.4.35}$$

Now we estimate the term B_2. From the definition of F we get,

$$\begin{aligned} B_2 \le cT^{1/2}\big[&\|u_1v_1v_{2x}\|_{L^2_xL^2_T} + \|u_1v_2v_{1x}\|_{L^2_xL^2_T} + \|v_1v_2u_{1x}\|_{L^2_xL^2_T} + \|u_1v_2v_{2x}\|_{L^2_xL^2_T} \\ &+ \|u_2v_1v_{1x}\|_{L^2_xL^2_T} + \|u_2v_1v_{2x}\|_{L^2_xL^2_T} + \|u_2v_2v_{1x}\|_{L^2_xL^2_T} + \|v_1v_2u_{2x}\|_{L^2_xL^2_T} \\ &+ \|u_2v_2v_{2x}\|_{L^2_xL^2_T} + \|v_2^2u_{1x}\|_{L^2_xL^2_T} + \|v_1^2u_{2x}\|_{L^2_xL^2_T} + \|v_2^2u_{2x}\|_{L^2_xL^2_T}\big] \\ := B_{2,1} &+ B_{2,2} + \cdots + B_{2,12}. \end{aligned} \tag{3.4.36}$$

Using Hölder's inequality, Theorem 3.8, Theorem 3.9 and $T \sim N^{-(1-s)}$ we obtain,

$$\begin{aligned} B_{2,1} &\le cT^{1/2}\|u_1\|_{L^4_xL^\infty_T}\|v_1\|_{L^4_xL^\infty_T}\|v_{2x}\|_{L^\infty_xL^2_T} \\ &\le cT^{1/2}|\!|\!|u_1|\!|\!|_{1/4}|\!|\!|v_1|\!|\!|_{1/4}|\!|\!|v_2|\!|\!|_{1/4} \\ &\le cT^{1/2}|\!|\!|(u_1,v_1)|\!|\!|^2_{1/4}|\!|\!|(u_2,v_2)|\!|\!|_{1/4} \\ &\le cN^{-\frac{1}{2}(1-s)}N^{\frac{1}{2}(1-s)}N^{\frac{1}{4}-s} \le cN^{\rho-s}. \end{aligned}$$

We can have the similar estimates for $B_{2,j},\ j = 2,3,\ldots,12,$ so that

$$B_2 \le cN^{\rho-s}. \tag{3.4.37}$$

Also, with the argument applied in B_1 and B_2 we get the similar estimates for B_3 and B_4 respectively.

Therefore, using (3.4.35), (3.4.37) and the analogous estimates for B_3 and B_4 we obtain from (3.4.24),

$$|\!|\!|\!|(u_2,v_2)|\!|\!|\!|_\rho \le c\|(\phi_2,\psi_2)\|_{X^\rho} + \Big\{\frac{1}{2} + cN^{-\frac{3}{4}s}\Big\}|\!|\!|\!|(u_2,v_2)|\!|\!|\!|_\rho + cN^{\rho-s}.$$

Choosing $N \gg 1$ such that $cN^{-\frac{3}{4}s} < 1/3$ we get the required result. □

Proposition 3.1. *Define* $|\!|\!|(u_2,v_2)|\!|\!|_0 = \max\{|\!|\!|u_2|\!|\!|_0, |\!|\!|v_2|\!|\!|_0\}$ *where,*

$$|\!|\!|f|\!|\!|_0 = \|f\|_{L^\infty_TL^2_x} + \|\partial_xf\|_{L^\infty_xL^2_T} + \|f\|_{L^5_xL^{10}_T}.$$

Let (u_1,v_1) *and* (u_2,v_2) *be solutions to the IVPs (3.4.2) and (3.4.4) with* $(\phi_1,\psi_1) \in X^1$ *and* $(\phi_2,\psi_2) \in X^s$ *respectively satisfying* $\|(\phi_1,\psi_1)\|_{X^1} \sim N^{1-s}$ *and* $\|(\phi_2,\psi_2)\|_X \sim N^{-s}$, $1/4 \le s < 1$. *Then*

$$|\!|\!|(u_2,v_2)|\!|\!|_0 \sim N^{-s}. \tag{3.4.38}$$

Proof. By the definition of $\|\!|\cdot|\!\|_0$ and linear estimates established in section 3.3 we obtain,

$$\|\!|(u_2,v_2)|\!\|_0 \le c\|(\phi_2,\psi_2)\|_X + cT^{1/2}\{\|F\|_{L^2_xL^2_T} + \|G\|_{L^2_xL^2_T}\}. \tag{3.4.39}$$

From the definition of F we get,

$$\begin{aligned}\|F\|_{L^2_xL^2_T} \le c[&\|u_1v_1v_{2x}\|_{L^2_xL^2_T} + \|u_1v_2v_{1x}\|_{L^2_xL^2_T} + \|v_1v_2u_{1x}\|_{L^2_xL^2_T} + \|u_1v_2v_{2x}\|_{L^2_xL^2_T}\\ &+ \|u_2v_1v_{1_x}\|_{L^2_xL^2_T} + \|u_2v_1v_{2x}\|_{L^2_xL^2_T} + \|u_2v_2v_{1x}\|_{L^2_xL^2_T} + \|v_1v_2u_{2x}\|_{L^2_xL^2_T}\\ &+ \|u_2v_2v_{2x}\|_{L^2_xL^2_T} + \|v_2^2u_{1x}\|_{L^2_xL^2_T} + \|v_1^2u_{2x}\|_{L^2_xL^2_T} + \|v_2^2u_{2x}\|_{L^2_xL^2_T}]\\ :=\ & F_1 + F_2 + \cdots + F_{12}.\end{aligned} \tag{3.4.40}$$

Now, using Hölder's inequality and the definition of $\|\!|\cdot|\!\|_0$ and $\|\!|\cdot|\!\|_{1/4}$ we obtain,

$$F_1 \le c\|u_1\|_{L^4_xL^\infty_T}\|v_1\|_{L^4_xL^\infty_T}\|v_{2x}\|_{L^\infty_xL^2_T} \le c\|\!|u_1|\!\|_{1/4}\|\!|v_1|\!\|_{1/4}\|\!|v_2|\!\|_0 \le c\|\!|(u_1,v_1)|\!\|_{1/4}^2\|\!|(u_2,v_2)|\!\|_0.$$

Similarly,

$$F_2 \le c\|u_1\|_{L^4_xL^\infty_T}\|u_1v_2\|_{L^5_xL^{10}_T}\|v_{1x}\|_{L^{20}_xL^{5/2}_T} \le c\|\!|u_1|\!\|_{1/4}\|\!|v_2|\!\|_0\|\!|v_1|\!\|_{1/4} \le c\|\!|(u_1,v_1)|\!\|_{1/4}^2\|\!|(u_2,v_2)|\!\|_0.$$

We can apply the similar argument to get,

$$\begin{aligned}&F_j \le c\|\!|(u_1,v_1)|\!\|_{1/4}^2\|\!|(u_2,v_2)|\!\|_0, \quad j = 3,5,11.\\ &F_j \le c\|\!|(u_1,v_1)|\!\|_{1/4}\|\!|(u_2,v_2)|\!\|_{1/4}\|\!|(u_2,v_2)|\!\|_0, \quad j = 4,6,7,8,10.\\ &F_j \le c\|\!|(u_2,v_2)|\!\|_{1/4}^2\|\!|(u_2,v_2)|\!\|_0, \quad j = 9,12.\end{aligned}$$

Also, we have the similar estimates for $\|G\|_{L^2_xL^2_T}$. Collecting all these estimates we get from (3.4.39),

$$\begin{aligned}\|\!|(u_2,v_2)|\!\|_0 \le c\|(\phi_2,\psi_2)\|_X + cT^{1/2}\big\{&\|\!|(u_1,v_1)|\!\|_{1/4}^2\\ &+ \|\!|(u_1,v_1)|\!\|_{1/4}\|\!|(u_2,v_2)|\!\|_{1/4} + \|\!|(u_2,v_2)|\!\|_{1/4}^2\big\}\|\!|(u_2,v_2)|\!\|_0.\end{aligned} \tag{3.4.41}$$

Finally, considering $cT^{1/2}\{\|\!|(u_1,v_1)\|\!|_{1/4}^2 + \|\!|(u_1,v_1)\|\!|_{1/4}\|\!|(u_2,v_2)\|\!|_{1/4} + \|\!|(u_2,v_2)\|\!|_{1/4}^2\} < 3/10$ for the choice of T in Theorem 3.9 we get from (3.4.41),

$$\|\!|(u_2,v_2)\|\!|_0 \leq c\|(\phi_2,\psi_2)\|_X,$$

which gives the required result. □

Proposition 3.2. *If* (u_1,v_1) *is a solution to the IVP (3.4.2) in* $[0,T]$, $T<1$, *then*

$$\|u_1\|_{L_x^{4/(1+\theta)}L_T^\infty} + \|v_1\|_{L_x^{4/(1+\theta)}L_T^\infty} \leq c\|\!|(u_1,v_1)\|\!|_{(1+2\theta)/4}, \qquad 0\leq\theta<1 \tag{3.4.42}$$

and

$$\|u_{1x}\|_{L_x^{40/(20\theta-3)}L_T^{5/2}} + \|v_{1x}\|_{L_x^{40/(20\theta-3)}L_T^{5/2}} \leq cT^{\theta/2-1/8}\|\!|(u_1,v_1)\|\!|_\theta, \;\; 1/4\leq\theta\leq1. \tag{3.4.43}$$

Moreover, the solution (u_2,v_2) *to the IVP (3.4.4) satisfies,*

$$\|u_2\|_{L_x^{4/(1+\theta)}L_T^\infty} + \|v_2\|_{L_x^{4/(1+\theta)}L_T^\infty} \leq c\|\!|\!|(u_2,v_2)\|\!|\!|_{(1+2\theta)/4}, \qquad 0\leq\theta<1 \tag{3.4.44}$$

and

$$\|u_{2x}\|_{L_x^{40/(20\theta-3)}L_T^{5/2}} + \|v_{2x}\|_{L_x^{40/(20\theta-3)}L_T^{5/2}} \leq cT^{\theta/2-1/8}\|\!|\!|(u_2,v_2)\|\!|\!|_\theta, \quad 1/4\leq\theta\leq1. \tag{3.4.45}$$

Proof. The estimate (3.4.42) follows by using the equivalent integral formula for (u_1,v_1), the estimate (3.3.9) and the choice of T in the local well-posedness result. The estimate (3.4.43) follows by using the similar argument along with the estimate (3.3.13). The other estimates follow analogously. □

The following Proposition gives the estimates for the X^1 and X norms of the inhomogeneous part of the evolution of the high frequency part.

Proposition 3.3. *Let* F *and* G *be given by (3.4.5) and (3.4.6) with* (u_1,v_1) *and* (u_2,v_2) *solutions to the IVPs (3.4.2) and (3.4.4) respectively. Define,*

$$(z_1(t),z_2(t)) = \left(-\int_0^t U(t-t')F(t')\,dt',\; -\int_0^t U(t-t')G(t')\,dt'\right). \tag{3.4.46}$$

Let (ϕ_1, ψ_1) *and* (ϕ_2, ψ_2) *satisfy the hypothesis of Corollary 3.1 and Proposition 3.1. If* $3/5 < s < 1$, *then*

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X^1} \leq cN^{\frac{3-5s}{2}} \tag{3.4.47}$$

and

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X} \leq cN^{-s}. \tag{3.4.48}$$

Proof. Applying (3.3.3) and the definition of F we get,

$$\begin{aligned}
\|D_x z_1\|_{L^2} =& \ \|D_x \int_0^t U(t-t')F(t')\,dt'\|_{L^2} \leq c\|F\|_{L^1_x L^2_T} \\
\leq& \ c\big\{\|u_1v_1v_{2x}\|_{L^1_xL^2_T} + \|u_1v_2v_{1x}\|_{L^1_xL^2_T} + \|v_1v_2u_{1x}\|_{L^1_xL^2_T} + \|u_1v_2v_{2x}\|_{L^1_xL^2_T} \\
&+ \|u_2v_1v_{1x}\|_{L^1_xL^2_T} + \|u_2v_1v_{2x}\|_{L^1_xL^2_T} + \|u_2v_2v_{1x}\|_{L^1_xL^2_T} + \|v_1v_2u_{2x}\|_{L^1_xL^2_T} \\
&+ \|u_2v_2v_{2x}\|_{L^1_xL^2_T} + \|v_2^2u_{1x}\|_{L^1_xL^2_T} + \|v_1^2u_{2x}\|_{L^1_xL^2_T} + \|v_2^2u_{2x}\|_{L^1_xL^2_T}\big\} \\
:=& \ z_{1,1} + z_{1,2} + \cdots + z_{1,12}.
\end{aligned} \tag{3.4.49}$$

Now we estimate $z_{1,j}, j = 1, 2, \ldots, 12$. To get estimates for all these terms we use similar argument utilizing Theorem 3.8, Corollary 3.1, Proposition 3.1, Proposition 3.2 and the choice of T. For the sake of clarity let us consider the most difficult terms $u_1v_1v_{2x}$ and $u_1v_2v_{1x}$ in F and obtain,

$$\begin{aligned}
z_{1,1} =& \ c\|u_1v_1v_{2x}\|_{L^1_xL^2_T} \\
\leq& \ c\|u_1\|_{L^2_xL^\infty_T}\|v_1\|_{L^2_xL^\infty_T}\|v_{2x}\|_{L^\infty_xL^2_T} \\
\leq& \ c\|\!|(u_1, v_1)|\!\|_{3/4}^2 \|\!|(u_2, v_2)|\!\|_0 \\
\leq& \ cN^{\frac{3-5s}{2}}
\end{aligned}$$

and

$$\begin{aligned}
z_{1,2} =& \ c\|u_1v_2v_{1x}\|_{L^1_xL^2_T} \\
\leq& \ c\|u_1\|_{L^{8/3}_xL^\infty_T}\|v_2\|_{L^5_xL^{10}_T}\|v_{1x}\|_{L^{40/17}_xL^{5/2}_T} \\
\leq& \ cT^{3/8}\|\!|(u_1, v_1)|\!\|_{1/2}\|\!|(u_2, v_2)|\!\|_0\|\!|(u_1, v_1)|\!\|_1 \\
\leq& \ cN^{\frac{9-17s}{8}} \leq cN^{\frac{3-5s}{2}}.
\end{aligned}$$

We can obtain similar estimates for the other terms in (3.4.49) too.

Using an analogous argument we can get,

$$\|z_1\|_{L^2_x} \leq cN^{-s}.$$

Finally, we can also obtain similar estimates for z_2 and that concludes the proof. $\square$

Next, we derive some estimates that will be useful in the second step of the proof of the main result. Now, we consider the given data in X^s, $1/4 \leq s \leq s_1 < 1$, and split into low and high frequency parts according to the formula (3.4.1). For $s_0 := s_1 + \epsilon < 1$, the low frequency part $(\phi_1, \psi_1) \in X^{s_0}$ with $\|(\phi_1, \psi_1)\|_{X^{s_0}} \leq cN^{s_0 - s}$ and the high frequency part $(\phi_2, \psi_2) \in X^\rho$, $0 \leq \rho \leq s < s_0$, with $\|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho - s}$. Moreover, $\|(\phi_1, \psi_1)\|_X \leq c$ and by interpolation we obtain $\|(\phi_1, \psi_1)\|_{X^\beta} \leq cN^{\frac{\beta}{s_0}(s_0 - s)}$, $0 \leq \beta \leq s_0$. As earlier we evolve (ϕ_1, ψ_1) and (ϕ_2, ψ_2) according to the IVPs (3.4.2) and (3.4.4) respectively. In this case also, we can have the results analogous to Theorem 3.8 and Theorem 3.9 with the local existence time replaced by

$$T \sim \|(\phi_1, \psi_1)\|_{X^{1/4}}^{-4} \sim N^{-\frac{1}{s_0}(s_0 - s)}$$

and the estimate (3.4.9) replaced by

$$\vert\!\vert\!\vert(u_1, v_1)\vert\!\vert\!\vert_{X^\beta} \leq cN^{\frac{\beta}{s_0}(s_0 - s)}, \qquad 1/4 \leq \beta \leq s_0. \tag{3.4.50}$$

Also, it is easy to obtain the results analogous to Corollary 3.1 and Proposition 3.1, i.e.,

$$\vert\!\vert\!\vert\!\vert(u_2, v_2)\vert\!\vert\!\vert\!\vert_\rho \leq cN^{\rho - s}, \qquad 1/4 < \rho \leq s < s_0, \tag{3.4.51}$$

$$\vert\!\vert\!\vert(u_2, v_2)\vert\!\vert\!\vert_0 \leq cN^{-s}. \tag{3.4.52}$$

Our next result is similar to Proposition 3.3. Before establishing it let us explain in brief the argument we are going to employ. We want to develop an iteration process in X^{s_0} by incorporating the inhomogeneous part of the evolution of the high frequency part with the evolution of the low frequency part. For this, we need to know the growth of the X^{s_0} and X norms of the inhomogeneous part (z_1, z_2), given by (3.4.46), of the solution to the IVP (3.4.4).

For simplicity we analyze $z_1(t)$ considering one of the worst terms $u_1v_1v_{2x}$ in F and get estimate for

$$\|D_x^{s_0} z_1\|_{L^2} = \|D_x^{s_0} \int_0^t U(t-t')(u_1v_1v_{2x})(t')\, dt'\|_{L^2}.$$

Using (3.3.3) we get,

$$\|D_x^{s_0} z_1\|_{L^2} \leq cT^{\frac{1}{2}(1-s_0)} \|u_1v_1v_{2x}\|_{L_x^{2/(1+s_0)} L_T^2}. \tag{3.4.53}$$

Applying Hölder's inequality, Proposition 3.1 and Proposition 3.2 we get from (3.4.53),

$$\begin{aligned} \|D^{s_0} z_1\|_{L^2} &\leq cT^{\frac{1}{2}(1-s_0)} \|u_1\|_{L_x^{4/(1+s_0)} L_T^\infty} \|v_1\|_{L_x^{4/(1+s_0)} L_T^\infty} \|v_{2x}\|_{L_x^\infty L_T^2} \\ &\leq cT^{\frac{1}{2}(1-s_0)} \|\!|(u_1, v_1)|\!\|^2_{(1+2s_0)/4} \|\!|(u_2, v_2)|\!\|_0. \end{aligned} \tag{3.4.54}$$

We will get the same estimate if we consider z_2 too.

Note that we have control on $\|D^{s_0}(u_1, v_1)\|_{L^2}$ and are interested to control $\|D^{s_0}(z_1, z_2)\|_{L^2}$ by it. From (3.4.54) it is clear that we will have such control on $\|D^{s_0}(z_1, z_2)\|_{L^2}$ only if

$$s_0 \geq \frac{1+2s_0}{4}, \qquad \text{i.e.} \quad s_0 \geq 1/2, \tag{3.4.55}$$

which is true, since in the second step of the proof of the main result we take $s_0 \in (3/5, 1)$ (see proof of Theorem 3.2). This condition also implies that the number of derivatives of (z_1, z_2) is not less than those of (u_1, v_1) justifying the incorporation of (z_1, z_2) with (u_1, v_1) in the iteration process.

The following Proposition provides the estimates for the X^{s_0} and X norms of (z_1, z_2).

Proposition 3.4. *Let z_1 and z_2 be defined in* (3.4.46) *with F and G given by (3.4.5) and (3.4.6) respectively. Let (ϕ_1, ψ_1) and (ϕ_2, ψ_2) satisfy the respective hypotheses of Corollary 3.1 and*

Proposition 3.1, then for $s_0 \geq 1/2$*,*

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X^{s_0}} \leq cN^{\frac{3s_0-5s}{2}} \tag{3.4.56}$$

and

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X} \leq cN^{-s}. \tag{3.4.57}$$

Proof. Using (3.3.3) and the definition of F we get,

$$\begin{aligned}
\|D_x^{s_0} z_1\|_{L^2} =\ & \|D_x^{s_0}\int_0^t U(t-t')F(t')\,dt'\|_{L^2} \leq cT^{\frac{1}{2}(1-s_0)}\|F\|_{L_x^{2/(1+s_0)}L_T^2}\\
\leq\ & cT^{\frac{1}{2}(1-s_0)}\Big\{\|u_1v_1v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_1v_2v_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_1v_2u_{1x}\|_{L_x^{2/(1+s_0)}L_T^2}\\
& + \|u_1v_2v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_2v_1v_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_2v_1v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2}\\
& + \|u_2v_2v_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_1v_2u_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_2v_2v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2}\\
& + \|v_2^2u_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_1^2u_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_2^2u_{2x}\|_{L_x^{2/(1+s_0)}L_T^2}\Big\}\\
:=\ & z_{1,1} + z_{1,2} + \cdots + z_{1,12}.
\end{aligned} \tag{3.4.58}$$

Now we estimate $z_{1,j}, j = 1, 2, \ldots, 12$. To get estimates for all these terms we use similar argument utilizing (3.4.50), (3.4.51), (3.4.52), Proposition 3.2 and the choice of T. For the sake of clarity let us consider one of the most difficult terms $u_1v_1v_{2x}$ in F and obtain,

$$\begin{aligned}
z_{1,1} &= cT^{\frac{1}{2}(1-s_0)}\|u_1v_1v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2}\\
&\leq cT^{\frac{1}{2}(1-s_0)}\|u_1\|_{L_x^{4/(1+s_0)}L_T^\infty}\|v_1\|_{L_x^{4/(1+s_0)}L_T^\infty}\|v_{2x}\|_{L_x^\infty L_T^2}\\
&\leq cT^{\frac{1}{2}(1-s_0)}|\!|\!|(u_1, v_1)|\!|\!|^2_{(1+2s_0)/4}|\!|\!|(u_2, v_2)|\!|\!|_0\\
&\leq cN^{-\frac{1}{2}(1-s_0)\frac{1}{s_0}(s_0-s)}N^{\frac{1+2s_0}{2}\frac{1}{s_0}(s_0-s)}N^{-s}\\
&\leq cN^{\frac{3s_0-5s}{2}}.
\end{aligned}$$

Similar estimates can also be obtained for the other terms in (3.4.58).

An analogous argument leads to,

$$\|z_1\|_{L_x^2} \leq cN^{-s}.$$

Finally, we can also obtain similar estimates for z_2 and that concludes the proof. $\square$

3.5 Proof of the Global Well-posedness Result

In this section we give the proof of Theorem 3.2, the main result of this chapter.

Proof of Theorem 3.2: As mentioned in the introduction we carry-out the proof in two steps.

First step: Let $(\phi, \psi) \in X^s(\mathbb{R})$, $3/5 < s < 1$ and $N \gg 1$ be arbitrary but fixed. Let us decompose the initial data as in (3.4.1) to

$$\begin{cases} \phi(x) = \phi_1(x) + \phi_2(x), \\ \psi(x) = \psi_1(x) + \psi_2(x). \end{cases} \tag{3.5.1}$$

Then we have,

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \leq c \\ \|(\phi_1, \psi_1)\|_{X^\beta} \leq cN^{\beta(1-s)}, \qquad 0 \leq \beta \leq 1. \end{cases} \tag{3.5.2}$$

$$\|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho - s}, \quad 0 \leq \rho \leq s < 1. \tag{3.5.3}$$

Consider the IVP (3.4.2) with initial data $(\phi_1, \psi_1) \in X^\beta$, $1/4 \leq \beta \leq 1$. From Theorem 3.1 there exists T_0 satisfying

$$T_0 \leq c\|(\phi_1, \psi_1)\|_{X^{1/4}}^{-4} \sim cN^{-(1-s)}, \tag{3.5.4}$$

such that the IVP (3.4.2) has a unique solution (u_1, v_1) in the interval $[0, T_0]$. Moreover,

$$\sup_{t \in [0, T_0]} \|(u_1(t), v_1(t))\|_{X^1} \leq cN^{(1-s)}. \tag{3.5.5}$$

Now, we consider the IVP (3.4.4) with initial data (ϕ_2, ψ_2). In Theorem 3.9 we found that the IVP (3.4.4) has a unique solution (u_2, v_2) defined in the same interval of existence of the solution (u_1, v_1), $[0, T_0]$ and is given by (4.4.17), i.e.

$$\begin{cases} u_2(t) = U(t)\phi_2 + z_1(t) \\ v_2(t) = U(t)\psi_2 + z_2(t). \end{cases} \tag{3.5.6}$$

where $z_1(t)$ and $z_2(t)$ are given by (3.4.46).

As mentioned in the introduction, $u = u_1 + u_2$ and $v = v_1 + v_2$ solve the IVP (1.3.63) in the time interval $[0, T_0]$.

Given $T > 0$ arbitrary, we are interested in extending the solution (u, v) of the IVP (1.3.63) to the interval $[0, T]$. For this, we iterate the above process in each interval of size T_0 unless covering the whole interval. Now, at the time $t = T_0$ we have,

$$\begin{cases} u(T_0) = u_1(T_0) + U(T_0)\phi_2 + z_1(T_0) \\ v(T_0) = v_1(T_0) + U(T_0)\psi_2 + z_2(T_0). \end{cases} \tag{3.5.7}$$

Now we decompose $(u(T_0), v(T_0))$ as,

$$\begin{cases} u(T_0) = \tilde{u}_1(T_0) + \tilde{u}_2(T_0) \\ v(T_0) = \tilde{v}_1(T_0) + \tilde{v}_2(T_0), \end{cases} \tag{3.5.8}$$

where,

$$\begin{cases} \tilde{u}_1(T_0) = u_1(T_0) + z_1(T_0), \qquad \tilde{u}_2(T_0) = U(T_0)\phi_2 \\ \tilde{v}_1(T_0) = v_1(T_0) + z_2(T_0), \qquad \tilde{v}_2(T_0) = U(T_0)\psi_2, \end{cases} \tag{3.5.9}$$

and evolve $(\tilde{u}_1(T_0), \tilde{v}_1(T_0))$ and $(\tilde{u}_2(T_0), \tilde{v}_2(T_0))$ according to the IVPs (3.4.2) and (3.4.4) respectively. Using previous procedure, to get solution to the IVP (1.3.63) in $[T_0, 2T_0]$ we must guarantee that $(\tilde{u}_1(T_0), \tilde{v}_1(T_0))$ and $(\tilde{u}_2(T_0), \tilde{v}_2(T_0))$ satisfy the respective conditions (3.5.2) and (3.5.3).

Since $U(t)$ is unitary in H^ρ, $(\tilde{u}_2(T_0), \tilde{v}_2(T_0))$ satisfies the same growth condition as that of (ϕ_2, ψ_2), i.e, $(\tilde{u}_2(T_0), \tilde{v}_2(T_0)) \in X^\rho$ and

$$\|(\tilde{u}_2(T_0), \tilde{v}_2(T_0))\|_{X^\rho} = \|(\phi_2, \psi_2)\|_{X^\rho} \le cN^{\rho - s}, \quad \rho \le s.$$

Now, let us check how is the growth of the X^1-norm and the X-norm of $(\tilde{u}_1(T_0), \tilde{v}_1(T_0))$. Using Proposition 3.3 and estimate (3.5.5) we get

$$\begin{aligned} \|(\tilde{u}_1(T_0), \tilde{v}_1(T_0))\|_{X^1} &\leq \|(u_1(T_0), v_1(T_0))\|_{X^1} + \|(z_1(T_0), z_2(T_0))\|_{X^1} \\ &\leq cN^{(1-s)} + cN^{\frac{3-5s}{2}}. \end{aligned} \tag{3.5.10}$$

On the other hand, using conservation law (1.3.61) and Proposition 3.3 we obtain,

$$\begin{aligned} \|(\tilde{u}_1(T_0), \tilde{v}_1(T_0))\|_{X} &\leq \|(u_1(T_0), v_1(T_0))\|_{X} + \|(z_1(T_0), z_2(T_0))\|_{X} \\ &\leq \|(\phi_1, \psi_1)\|_X + cN^{-s} \\ &\leq c, \qquad\qquad \text{for sufficiently large N.} \end{aligned} \tag{3.5.11}$$

So, from (3.5.10) it is clear that the solution to the IVP (1.3.63) can be extended to the interval $[T_0, 2T_0]$ if we can guarantee that $N^{\frac{3-5s}{2}} \leq cN^{(1-s)}$ for large N and some appropriate values of s. In what follows we select these values not only to guarantee this condition for a single iteration but to cover the whole interval $[0, T]$.

To cover the interval $[0, T]$ we must iterate the above process T/T_0 times. As seen earlier, in each iteration, there will be a contribution of $\|(z_1, z_2)\|_{X^1}$ and $\|(z_1, z_2)\|_X$. From (3.5.10) we see that the total contribution of $\|(z_1, z_2)\|_{X^1}$ to cover $[0, T]$ is, $(T/T_0)N^{\frac{3-5s}{2}}$.

Thus the X^1-norm of (z_1, z_2) will grow uniformly as $N^{(1-s)}$ on the interval $[0, T]$ if we have,

$$\frac{T}{T_0} N^{\frac{3-5s}{2}} < cN^{(1-s)}. \tag{3.5.12}$$

Now, using $T_0 \sim N^{-(1-s)}$ from (3.5.4) we see that (3.5.12) is equivalent to,

$$TN^{\frac{3-5s}{2}} < c. \tag{3.5.13}$$

Therefore, to guarantee (3.5.12) we must choose $N = N(T)$ satisfying

$$N(T) = T^{\frac{2}{5s-3}},$$

with $\frac{5s-3}{2} > 0$, i.e. $s > 3/5$.

Let us show, with this choice the X-norm is also of $O(1)$. We know from (3.5.11) that the total contribution of $\|(z_1, z_2)\|_X$ to cover the interval $[0, T]$ is $(T/T_0)N^{-s}$. Now, with the choice of N we get for $3/5 < s \leq 1$,

$$\frac{T}{T_0} N^{-s} \leq cTN^{1-s}N^{-s} \leq c,$$

as required.

Hence, we conclude that the IVP (1.3.63) has global solution whenever $s > 3/5$.

Also, it is easy to see that the solution can be written in the form

$$u(t) = U(t)\phi + w_1(t) \qquad \text{and} \qquad v(t) = U(t)\psi + w_2(t),$$

with

$$\sup_{[0,T]} \|w_j(t)\|_{H^1} \leq cT^{2(1-s)/(5s-3)}. \tag{3.5.14}$$

From (3.5.14) and the choice of N we can obtain the following upper bound for the solution to the IVP (1.3.63) in the X^s norm

$$\sup_{[0,T]} \|(u(t), v(t))\|_{X^s} \leq cN^{s(1-s)}, \qquad 3/5 < s < 1. \tag{3.5.15}$$

Second step: Let $(\phi_0, \psi_0) \in X^s(\mathbb{R})$, $1/4 \leq s < s_0$, $s_0 \in (3/5, 1)$ and $N \gg 1$ be arbitrary but fixed. Let us decompose the initial data as in (3.4.1) to low and high frequency parts, then we have,

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \leq c \\ \|(\phi_1, \psi_1)\|_{X^{s_0}} \leq cN^{s_0 - s} \end{cases} \tag{3.5.16}$$

and

$$\|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho - s}, \quad 0 \leq \rho \leq s. \tag{3.5.17}$$

Also, by interpolating the estimates in (3.5.16) we get,

$$\|(\phi_1, \psi_1)\|_{X^\beta} \le cN^{\frac{\beta}{s_0}(s_0-s)}, \qquad 0 \le \beta \le s_0. \tag{3.5.18}$$

Consider the IVP (3.4.2) with initial data $(\phi_1, \psi_1) \in X^{s_0}$. In the first step we saw that the solution to the IVP (3.4.2) for given data in X^{s_0} exists in any time interval $[0, T]$. Moreover, from (3.5.15) we have the following uniform bound for the X^{s_0}-norm of the solution,

$$\sup_{[0,T]} \|(u_1(t), v_1(t))\|_{X^{s_0}} \le cN^{s_0(1-s_0)}. \tag{3.5.19}$$

As mentioned earlier, in this step we develop an iteration process in the space X^{s_0}, $s_0 < 1$ to extend the local solution to any time interval $[0, T]$. From (3.5.16) we have $\|(\phi_1, \psi_1)\|_{X^{s_0}} \le cN^{s_0-s}$, so we expect that the evolution of (ϕ_1, ψ_1) i.e., $(u_1(t), v_1(t))$ in $[0, T]$ also satisfy the same growth condition. In other words, we want the following uniform bound

$$\sup_{[0,T]} \|(u_1(t), v_1(t))\|_{X^{s_0}} \le cN^{s_0(1-s_0)} \le cN^{s_0-s}. \tag{3.5.20}$$

But for the validity of (3.5.20) we must have

$$s_0(1 - s_0) \le s_0 - s, \quad \text{i.e.,} \quad s_0^2 \ge s. \tag{3.5.21}$$

Therefore, from here onwards we take s_0 satisfying (3.5.21) such that (3.5.20) is valid.

Now, we consider the IVP (3.4.4) with initial data (ϕ_2, ψ_2). From an analysis analogous to Theorem 3.9 we see that there exist T_0 satisfying

$$T_0 \le c\|(\phi_1, \psi_1)\|_{X^{1/4}}^{-4} \le cN^{-\frac{1}{s_0}(s_0-s)}, \tag{3.5.22}$$

and a unique solution (u_2, v_2) to the IVP (3.4.4) in the interval $[0, T_0]$ given by,

$$u_2(t) = U(t)\phi_2 + z_1(t), \quad v_2(t) = U(t)\psi_2 + z_2(t), \tag{3.5.23}$$

where $z_1(t)$ and $z_2(t)$ are as in (3.4.46).

In this case also $u = u_1+v_1$ and $v = u_2+v_2$ solve the IVP (1.3.63) in the time interval $[0, T_0]$.

Given $T > 0$ arbitrary, we are interested to extend the solution (u, v) of the IVP (1.3.63) to the interval $[0, T]$. For this, we will iterate the above process in each interval of size T_0 unless covering the whole interval. At the time $t = T_0$ we have,

$$\begin{cases} u(T_0) = u_1(T_0) + U(T_0)\phi_2 + z_1(T_0) \\ v(T_0) = v_1(T_0) + U(T_0)\psi_2 + z_2(T_0). \end{cases} \tag{3.5.24}$$

Now we decompose $(u(T_0), v(T_0))$ by the formula,

$$\begin{cases} u(T_0) = \tilde{u}_1(T_0) + \tilde{u}_2(T_0) \\ v(T_0) = \tilde{v}_1(T_0) + \tilde{v}_2(T_0) \end{cases} \tag{3.5.25}$$

where,

$$\begin{cases} \tilde{u}_1(T_0) = u_1(T_0) + z_1(T_0), \qquad \tilde{u}_2(T_0) = U(T_0)\phi_2 \\ \tilde{v}_1(T_0) = v_1(T_0) + z_2(T_0), \qquad \tilde{v}_2(T_0) = U(T_0)\psi_2. \end{cases} \tag{3.5.26}$$

To extend the solution (u, v) to $[T_0, 2T_0]$ we proceed as in the first step by evolving $(\tilde{u}_1, \tilde{v}_1)$ and $(\tilde{u}_2, \tilde{v}_2)$ according to the IVPs (3.4.2) and (3.4.4) respectively. For this we need to guarantee that $(\tilde{u}_1, \tilde{v}_1)$ and $(\tilde{u}_2, \tilde{v}_2)$ satisfy the conditions (3.5.16) and (3.5.17) respectively. Because of the group property, $(\tilde{u}_2, \tilde{v}_2)$ satisfies the desired condition, i.e.,

$$\|(\tilde{u}_2, \tilde{v}_2)\|_{X^\rho} \sim \|(\phi_2, \psi_2)\|_{X^\rho} \sim N^{\rho-s}, \qquad 1/4 \le \rho \le s < s_0. \tag{3.5.27}$$

As in (3.5.11), it is easy to show that the first condition in (3.5.16) holds. Now, let us move to check the X^{s_0}-norm of $(\tilde{u}_1, \tilde{v}_1)$.

Note that, from (3.5.20) and Proposition 3.4 we have,

$$\begin{cases} \|(u_1(T_0), v_1(T_0))\|_{X^{s_0}} \le cN^{s_0-s} \\ \|(z_1(T_0), z_2(T_0))\|_{X^{s_0}} \le cN^{\frac{3s_0-5s}{2}}. \end{cases} \tag{3.5.28}$$

Therefore, at $t = T_0$, from (3.5.26) and (3.5.28) we see that $(\tilde{u}_1, \tilde{v}_1)$ increases as

$$\|(\tilde{u}_1, \tilde{v}_1)\|_{X^{s_0}} \leq cN^{s_0-s} + cN^{\frac{3s_0-5s}{2}}. \tag{3.5.29}$$

So, from (3.5.29) it is clear that the solution to the IVP (1.3.63) can be extended to the interval $[T_0, 2T_0]$ if we can guarantee that $N^{\frac{3s_0-5s}{2}} \leq cN^{s_0-s}$ for large N and some appropriate values of s and s_0. In what follows we select these values not only to guarantee this condition for a single iteration but to cover the whole interval $[0, T]$.

To cover the interval $[0, T]$ we must iterate the above process T/T_0 times. As seen above, in each iteration, there will be a contribution of $\|(z_1, z_2)\|_{X^{s_0}}$. From (3.5.29) we see that the total contribution of $\|(z_1, z_2)\|_{X^{s_0}}$ to cover $[0, T]$ is $(T/T_0)N^{\frac{3s_0-5s}{2}}$.

Thus the X^{s_0}-norm of (z_1, z_2) will grow uniformly as N^{s_0-s} on the interval $[0, T]$ if we have,

$$\frac{T}{T_0}N^{\frac{3s_0-5s}{2}} < cN^{s_0-s}. \tag{3.5.30}$$

Now, using $T_0 \sim N^{-\frac{1}{s_0}(s_0-s)}$ from (3.5.22) we see that (3.5.30) is equivalent to,

$$TN^{\frac{1}{2s_0}(s_0-s)(2+s_0)-s} < c. \tag{3.5.31}$$

Therefore, to guarantee (3.5.30) we must choose $N = N(T)$ satisfying

$$N(T) = T^{2s_0/\{2ss_0-(s_0-s)(2+s_0)\}},$$

with $2ss_0 - (s_0 - s)(2 + s_0) > 0$, which in turn gives,

$$s > \frac{2 + s_0}{2 + 3s_0}s_0. \tag{3.5.32}$$

Thus we need to choose s_0 in such a way that the RHS of (3.5.32) is a minimum positive number. Taking into account the identity (3.5.21), we must have $s_0 > \max\{3/5, \sqrt{s}\}$. But, as

$1/4 \leq s \leq 3/5$, we need to select $s_0 > \sqrt{s}$. Now, using this selection in (3.5.32) we obtain $1 > s > 4/9$. Therefore, the IVP (1.3.63) has global solution if $s > 4/9$ and this completes the proof of the theorem. □

3.6 Comments

As observed in Tao [86], the high-low frequency technique introduced by Bourgain [13], although improves the global well-posedness of the IVP for data below energy space, it is not strong enough to go all down to the local well-posedness level. There is a new method recently introduced by Colliander et. al. in [24, 25], the so called I-method and almost conserved quantity, which is quite powerful to get global solution below energy spaces. In the case of the single KdV and the mKdV equations, the authors in [24, 25] proved the sharp global results both in the real line as well as the periodic cases. We believe that this method can be utilized to obtain sharp global results for the system we considered in this chapter too, but his has to be done.

Chapter 4

Coupled System of the cKdV Equations

4.1 Introduction

In this chapter we consider the IVP associated to the coupled system of the critical KdV (cKdV) equations (1.3.57). Our results for the system (1.3.57) are in the same spirit to that of the critical KdV equation obtained by Kenig, Ponce and Vega in [52]. We recall that the authors in [52] proved that the IVP associated to the critical KdV equation is globally well-posed for small data in $H^s(\mathbb{R})$, $s \geq 0$. To obtain this result, they used the sharp version of the smoothing effects of Kato type (see [46]) satisfied by the group associated to the linear problem, combined with the contraction mapping principle. Here, we will adapt the techniques introduced by [52] to deal the system and obtain results that are compatible to the single cKdV equation. Moreover, using the high-low frequency technique introduced by Bourgain [13] and further simplified by [28] we prove a global well-posedness result for given data that are not so small.

The main results of this chapter are as follows. Our first result is concerned with the L^2-well-posedness.

Theorem 4.1. *Let* $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. *Then there exists* $\delta > 0$ *such that for the initial data with* $\|(u_0, v_0)\|_{L^2 \times L^2} < \delta$, *the IVP* (1.3.57) *admits a unique solution* (u, v) *satisfying*

$$(u, v) \in C(\mathbb{R} : L^2(\mathbb{R}) \times L^2(\mathbb{R})), \tag{4.1.1}$$

$$\|\partial_x u\|_{L^\infty_x L^2_t} < \infty, \qquad \|\partial_x v\|_{L^\infty_x L^2_t} < \infty, \tag{4.1.2}$$

$$\|u\|_{L^5_x L^{10}_t} < \infty, \qquad \|v\|_{L^5_x L^{10}_t} < \infty. \tag{4.1.3}$$

Moreover, the map $(\tilde{u_0}, \tilde{v_0}) \mapsto (\tilde{u}, \tilde{v})$ *from* $\{(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) : \|(u_0, v_0)\|_{L^2 \times L^2} < \delta\}$ *into the class defined by (4.1.1) to (4.1.3) is Lipschitz.*

The second result deals with the H^s-well-posedness, where $s > 0$.

Theorem 4.2. *Let* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$*,* $s > 0$*. There exists* $\delta > 0$ *such that for* $\|(u_0, v_0)\|_{H^s \times H^s} < \delta$*, the IVP* (1.3.57) *admits a unique solution* (u, v) *satisfying*

$$(u, v) \in C(\mathbb{R} : H^s(\mathbb{R}) \times H^s(\mathbb{R})), \tag{4.1.4}$$

$$\|\partial_x u\|_{L^\infty_x L^2_t} < \infty, \qquad \|\partial_x v\|_{L^\infty_x L^2_t} < \infty, \tag{4.1.5}$$

$$\|D^s_x \partial_x u\|_{L^\infty_x L^2_t} < \infty, \qquad \|D^s_x \partial_x v\|_{L^\infty_x L^2_t} < \infty, \tag{4.1.6}$$

$$\|u\|_{L^5_x L^{10}_t} < \infty, \qquad \|v\|_{L^5_x L^{10}_t} < \infty, \tag{4.1.7}$$

$$\|D^s_x u\|_{L^5_x L^{10}_t} < \infty, \qquad \|D^s_x v\|_{L^5_x L^{10}_t} < \infty. \tag{4.1.8}$$

Moreover, the map $(\tilde{u_0}, \tilde{v_0}) \mapsto (\tilde{u}, \tilde{v})$ *from* $\{(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) : \|(u_0, v_0)\|_{H^s \times H^s} < \delta\}$ *into the class defined by (4.1.4) to (4.1.8) is Lipschitz.*

The previous theorems give global solutions to the IVP (1.3.57) for data with small $H^s \times H^s$ norm. The following theorem asserts well-posedness for the IVP (1.3.57) for arbitrary data in $H^s \times H^s$, but in this case we only have local solutions whose proven existence time depends on the norm as well as the position of the data as well.

Theorem 4.3. *Let* $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$*. There exist* $T = T(u_0, v_0) > 0$ *and a unique strong solution* (u, v) *to the IVP* (1.3.57) *satisfying*

$$(u, v) \in C([-T, T] : L^2(\mathbb{R}) \times L^2(\mathbb{R})), \tag{4.1.9}$$

$$\|\partial_x u\|_{L^\infty_x L^2_T} < \infty, \qquad \|\partial_x v\|_{L^\infty_x L^2_T} < \infty, \tag{4.1.10}$$

$$\|u\|_{L^5_x L^{10}_T} < \infty, \qquad \|v\|_{L^5_x L^{10}_T} < \infty. \tag{4.1.11}$$

Moreover, for any $T' \in (0, T)$, there exists a neighborhood $\mathcal{V}$ of (u_0, v_0) in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that the map $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ from $\mathcal{V}$ into the class defined by (4.1.9) to (4.1.11) with T' in place of T is Lipschitz.

If $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 0$, then the previous result extends to the class

$$(u, v) \in C([-T,\, T] : H^s(\mathbb{R}) \times H^s(\mathbb{R})),$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} < \infty, \qquad \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} < \infty,$$

in the above time interval $[-T,\, T]$.

For $s > 0$ existence time for solutions can be shown to depends only on the $H^s \times H^s$ norm of the data. More precisely, we have the following result.

Theorem 4.4. *Suppose $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 0$. Then there exist $T = T(\|u_0\|_{s,2}, \|v_0\|_{s,2}) > 0$ and a unique solution (u, v) to the IVP* (1.3.57) *satisfying*

$$(u, v) \in C([-T,\, T] : H^s(\mathbb{R}) \times H^s(\mathbb{R})), \tag{4.1.12}$$

$$\|\partial_x u\|_{L_x^\infty L_T^2} < \infty, \qquad \|\partial_x v\|_{L_x^\infty L_T^2} < \infty, \tag{4.1.13}$$

$$\|u\|_{L_x^5 L_T^{10}} < \infty, \qquad \|v\|_{L_x^5 L_T^{10}} < \infty, \tag{4.1.14}$$

$$\|D_x^s u\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} u\|_{L_x^5 L_T^{10}} < \infty, \qquad \|D_x^s v\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} v\|_{L_x^5 L_T^{10}} < \infty, \tag{4.1.15}$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x u\|_{L_x^\infty L_T^2} < \infty, \qquad \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x v\|_{L_x^\infty L_T^2} < \infty. \tag{4.1.16}$$

Moreover, for any $T' \in (0, T)$, there exists a neighborhood $\mathcal{V}$ of (u_0, v_0) in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ such that the map $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ from $\mathcal{V}$ into the class defined by (4.1.12) to (4.1.16) with T' in place of T is Lipschitz.

Our next interest is to extend the local solution obtained in Theorem 4.4. Note that, for given data $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with $\|(u_0, v_0)\|_{L^2 \times L^2} < \delta$, we have from Theorem 4.1, a global solution to the IVP (1.3.57). In [91], Weinstein proved the following Gagliardo-Nirenberg type inequality for $u \in H^1(\mathbb{R})$,

$$\frac{1}{3}\int u^6 \leq \frac{1}{(\int S^2)^2}\Big(\int u^2\Big)^2 \int u_x^2, \tag{4.1.17}$$

where S is the solitary wave solution for (1.3.57).

Now, using (4.1.17), the conserved quantities mentioned earlier and the fact that

$$\int u^3v^3 \leq \frac{1}{2}\Big(\int u^6 + v^6\Big),$$

one can obtain an *a priori* estimate for $\|(u,v)\|_{H^1\times H^1}$ provided

$$\|(u_0,v_0)\|_{L^2\times L^2} < \|(S,S)\|_{L^2\times L^2}. \tag{4.1.18}$$

This *a priori* estimate yields global well-posedness for the IVP (1.3.57) for initial data in $H^1(\mathbb{R})\times H^1(\mathbb{R})$ satisfying (4.1.18). This situation is similar to the one we discussed above for a single critical KdV equation. Hence, a natural goal is to obtain global solutions for data in $H^s(\mathbb{R})\times H^s(\mathbb{R})$, $s>0$ satisfying $\delta < \|(u_0,v_0)\|_{L^2\times L^2} < \|(S,S)\|_{L^2\times L^2}$. A partial result in this direction is our next theorem.

Theorem 4.5. *Let $(u_0,v_0)\in H^s(\mathbb{R})\times H^s(\mathbb{R})$, where $s>\frac{3}{4}$. Suppose that $\|(u_0,v_0)\|_{L^2\times L^2} < \|(S,S)\|_{L^2\times L^2}$. Then the unique solution to the IVP* (1.3.57) *given by Theorem 4.4 can be extended to any interval of time* $[0,T]$.

Using Duhamel's principle, we prove these theorems by considering the associated integral equation associated to the IVP (1.3.57), i.e,

$$\begin{cases} u(t) = U(t)u_0 - \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt' \\ v(t) = U(t)v_0 - \int_0^t U(t-t')\partial_x(u^3v^2)(t')\,dt'. \end{cases} \tag{4.1.19}$$

So, our interest will be to solve the system of integral equations (4.1.19). We use the contraction mapping principle in appropriate metric spaces to prove Theorems 4.1 - 4.4. While, to prove the global well-posedness result of Theorem 4.5, we use the frequency splitting argument introduced by Bourgain in [13].

This chapter is organized as follows. In Section 3.3 we record some preliminary estimates associated to the linear problem and other relevant results. In Section 4.3 we give a proof of the local well-posedness results and global well-posedness results for small data. Finally, a proof of the global well-posedness result for data not so small will be given in Section 4.4.

4.2 Preliminary Estimates

In this section we give some linear estimates associated to the IVP (1.3.57). These estimates are not new and can be found in the literature. Consequently, we just sketch the idea of the proof and mention the references where they can be found. Let $U(t)$ be the group generated by the operator ∂_x^3. First, let us state the smoothing effects.

Following is the double smoothing effect that one obtains for solutions to the non-homogeneous linear problem

$$\begin{cases} u_t + \partial_x^3 u = f, \\ u(x,0) = 0. \end{cases} \tag{4.2.1}$$

Lemma 4.1. *If $f \in L_x^1 L_t^2$ then*

$$\|\partial_x^2 \int_0^t U(t-t')f(\cdot,t')\,dt'\|_{L_x^\infty L_t^2} \le c\|f\|_{L_x^1 L_t^2}. \tag{4.2.2}$$

Proof. See [54, 52]. □

The proof of the following result, that is the chain rule for fractional derivatives, can be found in [52].

Lemma 4.2. *Let $\alpha \in (0,1)$. Let $p, p_1, p_2, q, q_2 \in (1,\infty)$, $q_1 \in (1,\infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|D_x^\alpha F(f)\|_{L_x^p L_T^q} \le c\|F'(f)\|_{L_x^{p_1} L_T^{q_1}} \|D_x^\alpha f\|_{L_x^{p_2} L_T^{q_2}}. \tag{4.2.3}$$

The next lemma is a Sobolev type inequality, known as Gagliardo-Nirenberg inequality, whose proof is given in [29].

Lemma 4.3. *Let q, r be any numbers satisfying $1 \leq q, r \leq \infty$ and let j, m be any integers satisfying $0 \leq j < m$. If $f \in C_0^m(\mathbb{R}^n)$, then*

$$\|D^j f\|_{L^p} \leq c\|D^m f\|_{L^r}^{\theta}\|f\|_{L^q}^{1-\theta}, \tag{4.2.4}$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-\theta)\frac{1}{q},$$

for all θ in the interval $\frac{j}{m} \leq \theta \leq 1$, and c depends only on n, m, j, q, r, θ.

Let us record the following result which plays crucial role in our argument to prove Theorem 4.3.

Lemma 4.4. *Let $u_0 \in L^2(\mathbb{R})$. Then for any $\epsilon > 0$, there exist $T = T(u_0; \epsilon) > 0$ and $\delta = \delta(u_0; \epsilon)$ such that if $\|u_0 - \tilde{u}_0\|_{L^2} < \delta$, then*

$$\|\partial_x U(t)\tilde{u}_0\|_{L_x^\infty L_T^2} < \epsilon \tag{4.2.5}$$

and

$$\|U(t)\tilde{u}_0\|_{L_x^5 L_T^{10}} < \epsilon. \tag{4.2.6}$$

Proof. We give details to obtain the estimate (4.2.5), the proof of (4.2.6) is similar. Using the linear estimate (3.3.1), if one takes $\delta < \epsilon/2c$, to show (4.2.5) it is enough to prove that

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} < \frac{\epsilon}{2}. \tag{4.2.7}$$

Let us take $w_0 \in \mathcal{S}(\mathbb{R})$ such that $\|u_0 - w_0\|_{L^2} < \epsilon/4c$. Now, using the estimate (3.3.1), Sobolev inequality and group property we get,

$$\begin{aligned}
\|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} &\leq \|\partial_x U(t)(u_0 - w_0)\|_{L_x^\infty L_T^2} + \|\partial_x U(t)w_0\|_{L_x^\infty L_T^2} \\
&\leq c\|u_0 - w_0\|_{L_x^2} + cT^{1/2}\|U(t)\partial_x w_0\|_{L_x^\infty L_T^\infty} \\
&\leq \frac{\epsilon}{4} + cT^{1/2}\|w_0\|_{2,2}.
\end{aligned} \tag{4.2.8}$$

Now choose T small enough such that $cT^{1/2}\|w_0\|_{2,2} < \epsilon/4$ to obtain (4.2.5). □

In sequel, we record an inequality which is crucial in the proof of Theorems 4.4 and 4.5 and can be found in [52]. We have,

$$\|f\|_{L^5_xL^{10}_T} \leq cT^{1/p}\|f\|_{L^5_xL^q_T} \leq cT^{s/3}\|D_t^{s/3}f\|_{L^5_xL^{10}_T}, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{10}, \quad q=q(s)\in(10,\infty), \tag{4.2.9}$$

where the first inequality follows from the Hölder's inequality and the second follows from the Gagliardo-Nirenberg inequality (4.2.4). Note that, we can get inequality (4.2.9) even for functions that are defined only in the interval $[-T,T]$. In this case one needs to extend them to the real line with zero values outside this interval to be able to define Fourier transform and hence fractional derivative in the time variable.

Before leaving this section we give some identities that will be useful in the proof of Theorem 4.4.

Lemma 4.5. *The following identities hold*

$$D_t^{s/3}U(t)u_0 = cD_x^sU(t)u_0 \tag{4.2.10}$$

and

$$D_t^{s/3}\int_{-\infty}^{\infty}U(t-t')g(t')\,dt' = cD_x^s\int_{-\infty}^{\infty}U(t-t')g(t')\,dt'. \tag{4.2.11}$$

Proof. The proof of the identities (4.2.10) and (4.2.11) follows easily by using a simple observation

$$\begin{aligned} U(t)u_0 &= c\int_{-\infty}^{\infty}e^{i(t\xi^3+x\xi)}\hat{u_0}(\xi)\,d\xi \\ &= c\int_{-\infty}^{\infty}e^{it\eta}e^{ix\eta^{1/3}}\frac{\hat{u_0}(\eta^{1/3})}{\eta^{2/3}}\,d\eta. \end{aligned} \tag{4.2.12}$$

□

4.3 Proof of the Local Results and Global Results for Small Data

Proof of Theorem 4.1: We will prove this theorem following the argument in [52]. Let us define a metric space,

$$\mathcal{X} = \{(u,v)\in C(\mathbb{R}:X(\mathbb{R})) : \|\!|(u,v)|\!\| < \infty\},$$

where

$$\|(u,v)\| = \max\{\|u\|, \|v\|\},$$

with

$$\|f\| = \|f\|_{L_t^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_t^2} + \|f\|_{L_x^5 L_t^{10}}. \tag{4.3.1}$$

For some $a > 0$, let $\mathcal{X}_a = \{(u,v) \in \mathcal{X} : \|(u,v)\| < a\}$ be a ball in $\mathcal{X}$.

Now, we define the following application,

$$\begin{cases} \Phi_{u_0}[u,v](t) := U(t)u_0 - \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt' \\ \Psi_{v_0}[u,v](t) := U(t)v_0 - \int_0^t U(t-t')\partial_x(u^3v^2)(t')\,dt'. \end{cases} \tag{4.3.2}$$

We show that, there are $a > 0$ and $\delta > 0$, such that the application $\Phi \times \Psi$ maps $\mathcal{X}_a$ into $\mathcal{X}_a$ and is a contraction.

Exploiting the symmetry of the system, we will only estimate the first component Φ. The estimates for the second component Ψ are similar.

Using the linear estimate (3.3.2) and Hölder's inequality we obtain,

$$\begin{aligned} \|\Phi\|_{L_x^2} &\le \|U(t)u_0\|_{L_x^2} + \|\int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L_x^2} \\ &\le c\|u_0\|_{L_x^2} + c\|u^2v^3\|_{L_x^1L_t^2} \\ &\le c\|u_0\|_{L_x^2} + c\|u\|_{L_x^5L_t^{10}}^2 \|v\|_{L_x^5L_t^{10}}^3 \\ &\le c\|(u_0,v_0)\|_{L_x^2\times L_x^2} + c\|(u,v)\|^5. \end{aligned} \tag{4.3.3}$$

Therefore,

$$\|\Phi\|_{L_t^\infty L_x^2} \le c\|(u_0,v_0)\|_{L_x^2\times L_x^2} + c\|(u,v)\|^5. \tag{4.3.4}$$

Similarly, using (3.3.1), (4.2.2) and Hölder's inequality, we get

$$\begin{aligned}
\|\partial_x\Phi\|_{L^\infty_x L^2_t} &\leq \|\partial_x U(t)u_0\|_{L^\infty_x L^2_t} + \|\partial_x \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L^\infty_x L^2_t} \\
&\leq c\|u_0\|_{L^2_x} + c\|u^2v^3\|_{L^1_x L^2_t} \\
&\leq c\|(u_0,v_0)\|_{L^2_x\times L^2_x} + c\|\!|(u,v)|\!\|^5.
\end{aligned} \tag{4.3.5}$$

Finally, the use of (3.3.10), (3.3.11) and Hölder's inequality yields,

$$\begin{aligned}
\|\Phi\|_{L^5_x L^{10}_t} &\leq \|U(t)u_0\|_{L^5_x L^{10}_t} + \|\int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L^5_x L^{10}_t} \\
&\leq c\|u_0\|_{L^2_x} + c\|\partial_x(u^2v^3)\|_{L^{5/4}_x L^{10/9}_t} \\
&\leq c\|u_0\|_{L^2_x} + c\|u^2v^2\partial_x v\|_{L^{5/4}_x L^{10/9}_t} + c\|uv^3\partial_x u\|_{L^{5/4}_x L^{10/9}_t} \\
&\leq c\|u_0\|_{L^2_x} + c\|u\|^2_{L^5_x L^{10}_t}\|v\|^2_{L^5_x L^{10}_t}\|\partial_x v\|_{L^\infty_x L^2_t} + c\|u\|_{L^5_x L^{10}_t}\|v\|^3_{L^5_x L^{10}_t}\|\partial_x u\|_{L^\infty_x L^2_t} \\
&\leq c\|(u_0,v_0)\|_{L^2_x\times L^2_x} + c\|\!|(u,v)|\!\|^5.
\end{aligned} \tag{4.3.6}$$

From (4.3.4), (4.3.5) and (4.3.6) we obtain,

$$\|\!|\Phi|\!\| \leq c\|(u_0,v_0)\|_{L^2\times L^2} + c\|\!|(u,v)|\!\|^5. \tag{4.3.7}$$

In an analogous manner we can get,

$$\|\!|\Psi|\!\| \leq c\|(u_0,v_0)\|_{L^2_x\times L^2_x} + c\|\!|(u,v)|\!\|^5. \tag{4.3.8}$$

Hence, for $(u,v)\in\mathcal{X}_a$,

$$\|\!|(\Phi,\Psi)|\!\| \leq c\|(u_0,v_0)\|_{L^2_x\times L^2_x} + c\|\!|(u,v)|\!\|^5 \leq c\delta + ca^5. \tag{4.3.9}$$

Let us choose δ such that $c(10c\delta)^4 \leq 1/2$ and $a\in(2c\delta,3c\delta)$. With these choices we get from (4.3.9),

$$\|\!|(\Phi,\Psi)|\!\| \leq \frac{a}{2}+\frac{a}{2}.$$

Hence, one gets that the application $\Phi \times \Psi$ maps the ball $\mathcal{X}_a$ into itself.

Now, we move to show that $\Phi \times \Psi$ is a contraction. For this, let $(u, v), (u_1, v_1) \in \mathcal{X}_a$. Using the argument employed to obtain (4.3.4), we get,

$$
\begin{aligned}
\|\partial_x(\Phi[u,v]-\Phi[u_1,v_1])\|_{L^\infty_x L^2_t} &= \\
&= \|\partial_x \int_0^t U(t-t')\partial_x(u^2v^3 - u_1^2v_1^3)(t')\,dt'\|_{L^\infty_x L^2_t} \\
&\leq c\|u^2v^3 - u_1^2v_1^3\|_{L^1_x L^2_t} \\
&\leq c\|v^3u(u-u_1)\|_{L^1_x L^2_t} + c\|v^3u_1(u-u_1)\|_{L^1_x L^2_t} + c\|u_1^2v^2(v-v_1)\|_{L^1_x L^2_t} \\
&\quad + c\|u_1^2vu_1(v-v_1)\|_{L^1_x L^2_t} + c\|u_1^2v_1^2(v-v_1)\|_{L^1_x L^2_t} \\
&\leq c\|\!|(u,v)\|\!|^4\|\!|u-u_1\|\!| + c\|\!|(u_1,v_1)\|\!|^2\|\!|(u,v)\|\!|^2\|\!|v-v_1\|\!| \\
&\quad + c\|\!|(u,v)\|\!|^3\|\!|(u_1,v_1)\|\!|\|\!|u-u_1\|\!| + c\|\!|(u_1,v_1)\|\!|^3\|\!|(u,v)\|\!|\|\!|v-v_1\|\!| \\
&\quad + c\|\!|(u_1,v_1)\|\!|^4\|\!|v-v_1\|\!| \\
&\leq 5ca^4\|\!|(u-u_1, v-v_1)\|\!|.
\end{aligned} \tag{4.3.10}
$$

Similarly, with the argument used in (4.3.6), one gets

$$
\begin{aligned}
&\|\Phi[u,v]-\Phi[u_1,v_1])\|_{L^5_x L^{10}_x} = \\
&\quad = \|\int_0^t U(t-t')\partial_x(u^2v^3 - u_1^2v_1^3)(t')\,dt'\|_{L^5_t L^{10}_x} \\
&\quad \leq c\|\partial_x(u^2v^3 - u_1^2v_1^3)\|_{L^{5/4}_x L^{10/9}_t} \\
&\quad \leq c\{\|uv^2\partial_x v(u-u_1)\|_{L^{5/4}_x L^{10/9}_t} + \|u_1v^2\partial_x v(u-u_1)\|_{L^{5/4}_x L^{10/9}_t} + \|u_1^2v\partial_x v(v-v_1)\|_{L^{5/4}_x L^{10/9}_t} \\
&\qquad + \|u_1^2v_1\partial_x v(v-v_1)\|_{L^{5/4}_x L^{10/9}_t} + \|u_1^2v_1^2\partial_x(v-v_1)\|_{L^{5/4}_x L^{10/9}_t} + \|v^3\partial_x u(u-u_1)\|_{L^{5/4}_x L^{10/9}_t} \\
&\qquad + \|u_1v^2\partial_x u(v-v_1)\|_{L^{5/4}_x L^{10/9}_t} + \|u_1vv_1\partial_x u(v-v_1)\|_{L^{5/4}_x L^{10/9}_t} \\
&\qquad + \|u_1\partial_x uv_1^2(v-v_1)\|_{L^{5/4}_x L^{10/9}_t} + \|u_1v_1^3\partial_x(u-u_1)\|_{L^{5/4}_x L^{10/9}_t}\} \\
&\quad =: c(A_1 + \cdots + A_{10}).
\end{aligned} \tag{4.3.11}
$$

One can get estimates for $A_1, \cdots, A_{10}$ using Hölder's inequality. For the sake of clarity let us present estimates for A_1, A_3 and A_7, the rest are analogous.

$$A_1 \leq c\|u\|_{L^5_x L^{10}_t}\|v^2\|_{L^{5/2}_x L^5_t}\|\partial_x v\|_{L^\infty_x L^2_t}\|u-u_1\|_{L^5_x L^{10}_t} \leq c\|\!|(u,v)\|\!|^4\|\!|u-u_1\|\!|. \tag{4.3.12}$$

$$A_3 \leq c\|u_1^2\|_{L^{5/2}_x L^5_t}\|v\|_{L^5_x L^{10}_t}\|\partial_x v\|_{L^\infty_x L^2_t}\|v-v_1\|_{L^5_x L^{10}_t} \leq c\|\!|(u_1,v_1)\|\!|^2\|\!|(u,v)\|\!|^2\|\!|v-v_1\|\!|. \tag{4.3.13}$$

$$A_7 \leq c\|u_1\|_{L^5_x L^{10}_t}\|v^2\|_{L^{5/2}_x L^5_t}\|\partial_x u\|_{L^\infty_x L^2_t}\|v-v_1\|_{L^5_x L^{10}_t} \leq c\|\!|(u_1,v_1)\|\!|\|\!|(u,v)\|\!|^3\|\!|v-v_1\|\!|. \tag{4.3.14}$$

Now, inserting estimates for $A_1, \cdots, A_{10}$ in (4.3.11) we get

$$\|\Phi[u,v]-\Phi[u_1,v_1])\|_{L^5_x L^{10}_x} \leq 10ca^4\|\!|(u-u_1, v-v_1)\|\!|. \tag{4.3.15}$$

Also, using the arguments employed to get estimates (4.3.5) and (4.3.10), it is easy to obtain

$$\|\Phi[u,v]-\Phi[u_1,v_1])\|_{L^\infty_t L^2_x} \leq 5ca^4\|\!|(u-u_1, v-v_1)\|\!|. \tag{4.3.16}$$

With an analogous argument we can obtain the similar estimates for Ψ too.

Combining all these estimates and the choice $c(10c\delta)^4 \leq 1/2$ we get

$$\|\!|(\Phi[u,v]-\Phi[u_1,v_1], \Psi[u,v]-\Psi[u_1,v_1])\|\!| \leq 1/2\|\!|(u-u_1, v-v_1)\|\!|. \tag{4.3.17}$$

Hence $(\Phi,\Psi): \mathfrak{X}_a \to \mathfrak{X}_a$ is a contraction. The rest of the proof follows a standard argument. $\square$

Now, we provide proof for the second main result of this chapyer.

Proof of Theorem 4.2. As in the proof of the previous theorem, for some $a>$, let us consider a ball

$$\mathfrak{X}^s_a = \{(u,v) \in C(\mathbb{R} : X^s(\mathbb{R})) : \|\!|(u,v)\|\!|_s < a\},$$

in the complete metric space

$$\mathcal{X}^s = \{(u,v) \in C(\mathbb{R} : X^s(\mathbb{R})) : \|\!|(u,v)\|\!|_s < \infty\},$$

where

$$\|\!|(u,v)\|\!|_s = \max\{\|\!|u\|\!|_s, \|\!|v\|\!|_s\},$$

with

$$\|\!|f\|\!|_s = \|D_x^s f\|_{L_t^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_t^2} + \|D_x^s \partial_x f\|_{L_x^\infty L_t^2} + \|f\|_{L_x^5 L_t^{10}} + \|D_x^s f\|_{L_x^5 L_t^{10}}. \quad (4.3.18)$$

Now our aim is to show that, there exist $a > 0$ and $\delta > 0$, such that the application $\Phi \times \Psi$ defined by (4.3.2) maps $\mathcal{X}_a^s$ into $\mathcal{X}_a^s$ and is a contraction.

Here also, using symmetry of the system we will only estimate the first component Φ.

The estimates for the norms $\|\partial_x \Phi\|_{L_x^\infty L_t^2}$ and $\|\Phi\|_{L_x^5 L_t^{10}}$ are already obtained in (4.3.5) and (4.3.6) respectively. Now, using the linear estimates established in Section 4.2 along with Leibniz rule and chain rule for fractional derivatives we obtain

$$\begin{aligned}
\|D_x^s \Phi\|_{L_x^2} &\leq \|D_x^s U(t)u_0\|_{L_x^2} + \|D_x^s \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L_x^2} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2v^3)\|_{L_x^1 L_t^2} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2v^3) - D_x^s(u^2)v^3 - u^2 D_x^s(v^3)\|_{L_x^1 L_t^2} \\
&\quad + c\|D_x^s(u^2)v^3\|_{L_x^1 L_t^2} + c\|u^2 D_x^s(v^3)\|_{L_x^1 L_t^2} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2)\|_{L_x^{5/2} L_t^5}\|v^3\|_{L_x^{5/3} L_t^{10/3}} + c\|u^2\|_{L_x^{5/2} L_t^5}\|D_x^s(v^3)\|_{L_x^{5/3} L_t^{10/3}} \\
&\leq c\|u_0\|_{H^s} + c\|u\|_{L_x^5 L_t^{10}}\|D_x^s u\|_{L_x^5 L_t^{10}}\|v\|_{L_x^5 L_t^{10}}^3 + c\|u\|_{L_x^5 L_t^{10}}^2\|D_x^s v\|_{L_x^5 L_t^{10}}\|v\|_{L_x^5 L_t^{10}}^2 \\
&\leq c\|(u_0,v_0)\|_{H^s \times H^s} + c\|\!|(u,v)\|\!|_s^5.
\end{aligned} \quad (4.3.19)$$

Therefore

$$\|D_x^s \Phi\|_{L_t^\infty L_x^2} \leq c\|(u_0,v_0)\|_{H^s \times H^s} + c\|\!|(u,v)\|\!|_s^5. \quad (4.3.20)$$

Similarly,

$$
\begin{aligned}
\|D_x^s\partial_x\Phi\|_{L_x^\infty L_t^2} &\leq \|D_x^s\partial_x U(t)u_0\|_{L_x^\infty L_t^2} + \|D_x^s\partial_x \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L_x^\infty L_t^2}\\
&\leq c\|D_x^s u_0\|_{L_x^2} + c\|\partial_x^2 \int_0^t U(t-t')D_x^s(u^2v^3)(t')\,dt'\|_{L_x^\infty L_t^2}\\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2v^3)\|_{L_x^1L_t^2}\\
&\leq c\|(u_0,v_0)\|_{H^s\times H^s} + c\vert\kern-0.25ex\vert\kern-0.25ex\vert(u,v)\vert\kern-0.25ex\vert\kern-0.25ex\vert_s^5,
\end{aligned}
\tag{4.3.21}
$$

and

$$
\begin{aligned}
\|D_x^s\Phi\|_{L_x^5L_t^{10}} &\leq \|D_x^s U(t)u_0\|_{L_x^5L_t^{10}} + \|D_x^s \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L_x^5L_t^{10}}\\
&\leq c\|D_x^s u_0\|_{L_x^2} + c\|D_x^s\partial_x(u^2v^3)\|_{L_x^{5/4}L_t^{10/9}}\\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2\partial_x v^3\|_{L_x^{5/4}L_t^{10/9}} + c\|D_x^s(\partial_x(u^2)v^3)\partial_x u\|_{L_x^{5/4}L_t^{10/9}}\\
&\leq c\|(u_0,v_0)\|_{H^s\times H^s} + c\vert\kern-0.25ex\vert\kern-0.25ex\vert(u,v)\vert\kern-0.25ex\vert\kern-0.25ex\vert_s^5.
\end{aligned}
\tag{4.3.22}
$$

Therefore, combining (4.3.5), (4.3.6) and (4.3.20) – (4.3.22) we obtain,

$$
\vert\kern-0.25ex\vert\kern-0.25ex\vert\Phi\vert\kern-0.25ex\vert\kern-0.25ex\vert_s \leq c\|(u_0,v_0)\|_{H^s\times H^s} + c\vert\kern-0.25ex\vert\kern-0.25ex\vert(u,v)\vert\kern-0.25ex\vert\kern-0.25ex\vert_s^5. \tag{4.3.23}
$$

In an analogous manner it is easy to get,

$$
\vert\kern-0.25ex\vert\kern-0.25ex\vert\Psi\vert\kern-0.25ex\vert\kern-0.25ex\vert_s \leq c\|(u_0,v_0)\|_{H^s\times H^s} + c\vert\kern-0.25ex\vert\kern-0.25ex\vert(u,v)\vert\kern-0.25ex\vert\kern-0.25ex\vert_s^5. \tag{4.3.24}
$$

Hence, for $(u,v)\in\mathcal{X}_a^s$,

$$
\vert\kern-0.25ex\vert\kern-0.25ex\vert(\Phi,\Psi)\vert\kern-0.25ex\vert\kern-0.25ex\vert_s \leq c\|(u_0,v_0)\|_{H^s\times H^s} + c\vert\kern-0.25ex\vert\kern-0.25ex\vert(u,v)\vert\kern-0.25ex\vert\kern-0.25ex\vert_s^5 \leq c\delta + ca^5. \tag{4.3.25}
$$

Let us choose δ such that $c(10c\delta)^4 \leq 1/2$ and $a \in (2c\delta, 3c\delta)$. With these choices we get from (4.3.25),

$$\|\!|(\Phi, \Psi)\|\!|_s \leq \frac{a}{2} + \frac{a}{2}.$$

Therefore, $\Phi \times \Psi$ maps $\mathcal{X}_a^s$ into $\mathcal{X}_a^s$.

With the similar argument, one can prove that $\Phi \times \Psi$ is a contraction. The rest of the proof follows standard argument. □

Remark 4.1. *In the Theorems 4.1 and 4.2 we obtained the global solution to the IVP* (1.3.57) *for given data with $H^s \times H^s$ norm less than δ. It would be interesting to find the exact value of δ. As a motivation for this, there is a similar result for the single critical KdV equation in the work of Angulo, Bona, Linares and Scialom in [4].*

Proof of Theorem 4.3. First, let us prove the theorem for given data in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Now define a complete metric space $\mathcal{X}^T$, in which we are going to find solution to the IVP (1.3.67), by

$$\mathcal{X}^T = \{(u, v) \in C([-T, T] : X(\mathbb{R})) : \|\!|(u, v)\|\!| < \infty\},$$

where

$$\|\!|(u, v)\|\!| = \max\{\|\!|u\|\!|, \|\!|v\|\!|\},$$

with

$$\|\!|u\|\!| = \|u - U(t)u_0\|_{L_T^\infty L_x^2} + \|\partial_x u\|_{L_x^\infty L_T^2} + \|u\|_{L_x^5 L_T^{10}} \tag{4.3.26}$$

and similar for v. Also, for some $a > 0$, define a ball

$$\mathcal{X}_a^T = \{(u, v) \in \mathcal{X}^T : \|\!|(u, v)\|\!| < a\}.$$

We show that there is $a > 0$, such that the application $\Phi \times \Psi$ defined by (4.3.2) maps $\mathcal{X}_a^T$ into $\mathcal{X}_a^T$ and is a contraction.

Now, using linear estimates and Lemma 4.4 we get,

$$\begin{aligned}
\|\Phi - U(t)u_0\|_{L_T^\infty L_x^2} &\leq \| \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L_T^\infty L_x^2} \\
&\leq c\|u^2v^3\|_{L_x^1 L_T^2} \leq c\|u\|^2_{L_x^5 L_T^{10}}\|v\|^3_{L_x^5 L_T^{10}} \\
&\leq c\|\!|(u,v)|\!\|^5,
\end{aligned} \tag{4.3.27}$$

$$\begin{aligned}
\|\partial_x\Phi\|_{L_x^\infty L_T^2} &\leq \|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} + \|\partial_x \int_0^t U(t-t')\partial_x(u^2v^3)(t')\,dt'\|_{L_x^\infty L_T^2} \\
&\leq c\epsilon + c\|u^2v^3\|_{L_x^1 L_T^2} \\
&\leq c\epsilon + c\|\!|(u,v)|\!\|^5
\end{aligned} \tag{4.3.28}$$

and similarly,

$$\|\Phi\|_{L_x^5 L_T^{10}} \leq c\epsilon + c\|\!|(u,v)|\!\|^5. \tag{4.3.29}$$

In an analogous manner we can obtain similar estimates for Ψ too.

Combining all these estimates we obtain for $(u,v) \in \mathcal{X}_a^T$,

$$\|\!|(\Phi,\Psi)|\!\| \leq c\epsilon + c\|\!|(u,v)|\!\|^5 \leq c\epsilon + ca^5. \tag{4.3.30}$$

If we choose ϵ and a in such a way that $c\epsilon + ca^4 < a$, we get $(\Phi,\Psi) \in \mathcal{X}_a^T$.

For $(u,v), (u_1,v_1) \in \mathcal{X}_a^T$, a similar argument leads to

$$\|\!|(\Phi[u,v]-\Phi[u_1,v_1], \Psi[u,v]-\Psi[u_1,v_1])|\!\| \leq 10ca^4\|\!|(u-u_1, v-v_1)|\!\|. \tag{4.3.31}$$

Thus, for $10ca^4 < 1/2$, $\Phi \times \Psi$ is a contraction on $\mathcal{X}_a^T$.

Note that, if we choose $\epsilon > 0$ such that $c(10c\epsilon)^4 < 1/2$ and $a \in (2c\epsilon, 3c\epsilon)$ then the both conditions $c\epsilon + ca^4 < a$ and $10ca^4 < 1/2$ are satisfied. A standard argument completes the rest of the proof in this case. The general case is similar, since as in the Proof of Theorem 4.2, the estimates in the involved norms appear linearly after interpolation. $\square$

Proof of Theorem 4.4. Following the procedure employed in the proof of the previous theorems, let us consider a ball

$$\mathcal{X}_a^T = \{(u,v) \in C([-T,T] : X^s(\mathbb{R})) : \|\!|(u,v)|\!\|_s < a\},$$

in a complete metric space, in which we are going to find solution to the IVP (1.3.67)

$$\mathcal{X}^T = \{(u,v) \in C([-T,T] : X^s(\mathbb{R})) : \|\!|(u,v)|\!\|_s < \infty\},$$

where

$$\|\!|(u,v)|\!\|_s = \max\{\|\!|u|\!\|_s, \|\!|v|\!\|_s\},$$

with

$$\begin{aligned} \|\!|f|\!\|_s = &\|D_s^s f\|_{L_T^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x f\|_{L_x^\infty L_T^2} + \|f\|_{L_x^5 L_T^{10}} + \|D_x^s f\|_{L_x^5 L_T^{10}} \\ &+ \|D_t^{s/3} f\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} \partial_x f\|_{L_x^\infty L_T^2}. \end{aligned} \tag{4.3.32}$$

Our aim is to show that, for some $a > 0$ and $T > 0$, the application $\Phi \times \Psi$ defined by (4.3.2) maps $\mathcal{X}_a^T$ into $\mathcal{X}_a^T$ and is a contraction.

As earlier, we will estimate only the first component Φ. Using the linear estimates established in section 4.2, Leibniz rule and chain rule for fractional derivatives along with the estimate (4.2.9) we obtain

$$\begin{aligned} \|D_x^s \Phi\|_{L_x^2} &\leq c\|u_0\|_{H^s} + c\|u\|_{L_x^5 L_T^{10}} \|D_x^s u\|_{L_x^5 L_T^{10}} \|v\|_{L_x^5 L_T^{10}}^3 + c\|u\|_{L_x^5 L_T^{10}}^2 \|D_x^s v\|_{L_x^5 L_T^{10}} \|v\|_{L_x^5 L_T^{10}}^2 \\ &\leq c\|u_0\|_{H^s} + cT^{s/3} \|D_t^{s/3} u\|_{L_x^5 L_T^{10}} \|D_x^s u\|_{L_x^5 L_T^{10}} T^s \|D_t^{s/3} v\|_{L_x^5 L_T^{10}}^3 \\ &\quad + cT^{2s/3} \|D_t^{s/3} u\|_{L_x^5 L_T^{10}}^2 \|D_x^s v\|_{L_x^5 L_T^{10}} T^{2s/3} \|D_t^{s/3} v\|_{L_x^5 L_T^{10}}^2 \\ &\leq c\|u_0\|_{H^s} + cT^{4s/3} \|\!|u|\!\|_s^2 \|\!|v|\!\|_s^3 + cT^{4s/3} \|\!|u|\!\|_s^2 \|\!|v|\!\|_s^3 \\ &\leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|\!|(u,v)|\!\|_s^5. \end{aligned}$$

Therefore,

$$\|D_x^s \Phi\|_{L_T^\infty L_x^2} \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|\!|(u,v)|\!\|_s^5. \tag{4.3.33}$$

Similarly,

$$\begin{aligned} \|\partial_x \Phi\|_{L_x^\infty L_T^2} &\leq c\|u_0\|_{H^s} + c\|u\|_{L_x^5 L_T^{10}}^2 \|v\|_{L_x^5 L_T^{10}}^3 \\ &\leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|\!|(u,v)|\!\|_s^5, \end{aligned} \tag{4.3.34}$$

and

$$\|D_x^s\partial_x\Phi\|_{L_x^\infty L_T^2} \leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}|\!|\!|(u,v)|\!|\!|_s^5. \tag{4.3.35}$$

Finally,

$$\begin{aligned}\|\Phi\|_{L_x^5L_T^{10}} &\leq c\|u_0\|_{H^s} + c\|u\|^2_{L_x^5L_T^{10}}\|v\|^2_{L_x^5L_T^{10}}\|\partial_x v\|_{L_x^\infty L_T^2} + c\|u\|_{L_x^5L_T^{10}}\|v\|^3_{L_x^5L_T^{10}}\|\partial_x u\|_{L_x^\infty L_T^2}\\ &\leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}|\!|\!|(u,v)|\!|\!|_s^5,\end{aligned} \tag{4.3.36}$$

and

$$\|D_x^s\Phi\|_{L_x^5L_T^{10}} \leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}|\!|\!|(u,v)|\!|\!|_s^5. \tag{4.3.37}$$

Now, using (4.2.10), (4.2.11) and the argument employed to get estimates for other norms we can get,

$$\|D_t^{s/3}\Phi\|_{L_x^5L_T^{10}} \leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}|\!|\!|(u,v)|\!|\!|_s^5 \tag{4.3.38}$$

and

$$\|D_t^{s/3}\partial_x\Phi\|_{L_x^\infty L_T^2} \leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}|\!|\!|(u,v)|\!|\!|_s^5. \tag{4.3.39}$$

From (4.3.33) to (4.3.39) we obtain,

$$|\!|\!|\Phi|\!|\!|_s \leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}|\!|\!|(u,v)|\!|\!|_s^5. \tag{4.3.40}$$

In an analogous manner one can easily get,

$$|\!|\!|\Psi|\!|\!|_s \leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}|\!|\!|(u,v)|\!|\!|_s^5. \tag{4.3.41}$$

Hence, for $(u,v) \in \mathcal{X}_a^T$,

$$\|(\Phi,\Psi)\|_s \leq c\|(u_0,v_0)\|_{H^s\times H^s} + cT^{4s/3}\|(u,v)\|_s^5 \leq c\delta + ca^5. \tag{4.3.42}$$

Let us choose $a = 2c\|(u_0,v_0)\|_{H^s\times H^s}$ and T such that $cT^{4s/3}a^4 < 1/2$. With these choices we get from (4.3.42),

$$\|(\Phi,\Psi)\|_s \leq \frac{a}{2} + \frac{a}{2}.$$

Therefore, $\Phi \times \Psi$ maps $\mathcal{X}_a^T$ into $\mathcal{X}_a^T$.

With the similar argument, one can prove that $\Phi \times \Psi$ is a contraction. The rest of the proof follows standard argument. □

Remark 4.2. *From the choice of a and T in the proof of Theorem 4.4 it is clear that the local existence time is given by*

$$T \leq c\|(u_0,v_0)\|_{H^s\times H^s}^{-3/s}. \tag{4.3.43}$$

4.4 Proof of the Global Result for not so Small Data

As mentioned in the introduction, we use the frequency splitting argument of Bourgain. In particular, we closely follow the scheme in [28]. We decompose the given data $(u_0,v_0) \in X^s$, $s<1$ to low and high frequency terms as,

$$\begin{cases} u_0(x) = (\chi_{\{|\xi|\leq N\}}\widehat{u_0}(\xi))^\vee(x) + (\chi_{\{|\xi|>N\}}\widehat{u_0}(\xi))^\vee(x) := \phi_1(x) + \phi_2(x) \\ v_0(x) = (\chi_{\{|\xi|\leq N\}}\widehat{v_0}(\xi))^\vee(x) + (\chi_{\{|\xi|>N\}}\widehat{v_0}(\xi))^\vee(x) := \psi_1(x) + \psi_2(x), \end{cases} \tag{4.4.1}$$

where $N \gg 1$ arbitrary but fixed for now, whose exact value will be selected later.

Then we have, $(\phi_1, \psi_1) \in X^\beta,\ 0 < \beta \leq 1$ and $(\phi_2, \psi_2) \in X^\rho, \quad 0 < \rho \leq s < 1$ with

$$\|(\phi_1, \psi_1)\|_{X^\beta} \lesssim N^{\beta - s} \lesssim N^{1-s}, \qquad \|(\phi_1, \psi_1)\|_X < \|(S, S)\|_X, \tag{4.4.2}$$

and

$$\|(\phi_2, \psi_2)\|_{X^\rho} \lesssim N^{\rho - s}, \quad 0 < \rho \leq s < 1. \tag{4.4.3}$$

We evolve (ϕ_1, ψ_1) according to the IVP

$$\begin{cases} u_{1t} + u_{1xxx} + (u_1^2 v_1^3)_x = 0 \\ v_{1t} + v_{2xxx} + (u_1^3 v_1^2)_x = 0 \\ u_1(x, 0) = \phi_1(x), \quad v_1(x, 0) = \psi_1(x), \end{cases} \tag{4.4.4}$$

which is the same as the IVP (1.3.67). We evolve (ϕ_2, ψ_2) according to the difference equation

$$\begin{cases} u_{2t} + u_{2xxx} + ((u_1 + u_2)^2 (v_1 + v_2)^3)_x - (u_1^2 v_1^3)_x = 0 \\ v_{2t} + v_{2xxx} + ((u_1 + u_2)^3 (v_1 + v_2^2))_x - (u_1^3 v_1^2)_x = 0 \\ u_2(x, 0) = \phi_2(x), \quad v_2(x, 0) = \psi_2(x), \end{cases} \tag{4.4.5}$$

with coefficients depending on the solution (u_1, v_1) to the IVP (3.4.2). It is clear that $u = u_1 + u_2$ and $v = v_1 + v_2$ solve the IVP (1.3.67). For simplicity, let us write (4.4.5) as

$$\begin{cases} u_{2t} + u_{2xxx} + \partial_x F = 0 \\ v_{2t} + v_{2xxx} + \partial_x G = 0 \\ u_2(x, 0) = \phi_2(x), \quad v_2(x, 0) = \psi_2(x), \end{cases} \tag{4.4.6}$$

where

$$\begin{aligned} F = {} & 3u_1^2 v_1^2 v_2 + 3u_1^2 v_1 v_2^2 + u_1^2 v_2^3 + 2u_1 u_2 v_1^3 + 6u_1 u_2 v_1^2 v_2 + 6u_1 u_2 v_1 v_2^2 \\ & + 2u_1 u_2 v_2^3 + u_2^2 v_1^3 + 3u_2^2 v_1^2 v_2 + 3u_2^2 v_1 v_2^2 + u_2^2 v_2^3, \end{aligned} \tag{4.4.7}$$

and

$$\begin{aligned} G = {} & 2u_1^3 v_1 v_2 + u_1^3 v_2^2 + 3u_1^2 u_2 v_1^2 + 6u_1^2 u_2 v_1 v_2 + 3u_1^2 u_2 v_2^2 + 3u_1 u_2^2 v_1^2 \\ & + 6u_1 u_2^2 v_1 v_2 + 3u_1 u_2^2 v_2^2 + u_2^3 v_1^2 + 2u_2^3 v_1 v_2 + u_2^3 v_2^2. \end{aligned} \tag{4.4.8}$$

Note that from Theorem 4.4 we have the existence result for the IVP (4.4.4). To get the existence result for the IVP (4.4.6) we need the following theorem.

Theorem 4.6. *Suppose the initial data* (ϕ_1, ψ_1) *of the IVP (4.4.4) satisfy*

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \le c \\ \|(\phi_1, \psi_1)\|_{X^1} \le cN^{1-s}. \end{cases} \tag{4.4.9}$$

Then for the existence time $T \sim c\|(\phi_1, \psi_1)\|_{X^1}^{-3} \sim cN^{-3(1-s)}$ *obtained in Theorem 4.4*

(i) The solution (u_1, v_1) *to the IVP (3.4.2) satisfies,*

$$\sup_t \|(u_1(t), v_1(t))\|_{X^1} = \sup_t [\|u_1(t)\|_{H^1} + \|v_1(t)\|_{H^1}] \le cN^{1-s}. \tag{4.4.10}$$

(ii) Moreover, for any $\beta \in (0, 1)$*, the solution* (u_1, v_1) *to the IVP (3.4.2) satisfies,*

$$|\!|\!|(u_1, v_1)|\!|\!|_\beta \sim N^{(1-s)\beta}, \tag{4.4.11}$$

where $|\!|\!|(u_1, v_1)|\!|\!|_\beta = \max\{|\!|\!|u_1|\!|\!|_\beta, |\!|\!|v_1|\!|\!|_\beta\}$ *and* $|\!|\!|f|\!|\!|_\beta$ *as in* (4.3.32).

Proof. The proof of (4.4.10) follows by using the conservation laws combined with the Gagliardo-Nirenberg inequality. The estimate (4.4.11) can be obtained by using the hypothesis (4.4.9) and the local well-posedness result. □

The following theorem provides the local existence result for the IVP (4.4.6) whose existence time coincides with that of (u_1, v_1).

Theorem 4.7. *Let* $(\phi_2, \psi_2) \in X^s$, $s > 0$ *and* (u_1, v_1) *be the unique solution given by Theorem 4.4 satisfying the conditions of Theorem 3.8. Then there exists a unique solution* (u_2, v_2) *to the IVP (4.4.6) in the same interval of existence of* (u_1, v_1), $[0, T]$ *such that,*

$$(u_2, v_2) \in C([-T, T] : H^s(\mathbb{R}) \times H^s(\mathbb{R})), \tag{4.4.12}$$

$$\|\partial_x u_2\|_{L_x^\infty L_T^2} < \infty, \qquad \|\partial_x v_2\|_{L_x^\infty L_T^2} < \infty, \tag{4.4.13}$$

$$\|u_2\|_{L_x^5 L_T^{10}} < \infty, \qquad \|v_2\|_{L_x^5 L_T^{10}} < \infty, \tag{4.4.14}$$

$$\|D_x^s u_2\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} u_2\|_{L_x^5 L_T^{10}} < \infty, \qquad \|D_x^s v_2\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} v_2\|_{L_x^5 L_T^{10}} < \infty, \tag{4.4.15}$$

$$\|D_x^s \partial_x u_2\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x u_2\|_{L_x^\infty L_T^2} < \infty, \qquad \|D_x^s \partial_x v_2\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x v_2\|_{L_x^\infty L_T^2} < \infty. \tag{4.4.16}$$

Proof. The proof of this theorem follows the same argument used to prove Theorem 4.4. As earlier, we consider the equivalent integral equation associated to the IVP (4.4.6),

$$\begin{cases} u_2(t) = U(t)\phi_2 - \int_0^t U(t-t')\partial_x F(t')\, dt' \\ v_2(t) = U(t)\psi_2 - \int_0^t U(t-t')\partial_x G(t')\, dt', \end{cases} \tag{4.4.17}$$

where F and G are defined in (4.4.7) and (4.4.8) respectively.

Let us define a ball

$$\mathcal{X}_a^T = \{(u_2, v_2) \in C([0,T] : X^s(\mathbb{R})) : \|\!|(u_2, v_2)|\!\|_s < a\},$$

in a complete metric space

$$\mathcal{X}^T = \{(u_2, v_2) \in C([0,T] : X^s(\mathbb{R})) : \|\!|(u_2, v_2)|\!\|_s < \infty\},$$

where

$$\|\!|(u_2, v_2)|\!\|_s = \max\{\|\!|u_2|\!\|_s, \|\!|v_2|\!\|_s\},$$

with $\|\!|f|\!\|_s$ as in (4.3.32).

Finally, we define,

$$\begin{cases} \Phi_{\phi_2}[u_2, v_2] = U(t)\phi_2 - \int_0^t U(t-t')\partial_x F(t')\, dt' \\ \Psi_{\psi_2}[u_2, v_2] = U(t)\psi_2 - \int_0^t U(t-t')\partial_x G(t')\, dt'. \end{cases} \tag{4.4.18}$$

and show that, for some $a > 0$ and $T > 0$, the application $\Phi \times \Psi$ maps $\mathcal{X}_a^T$ into $\mathcal{X}_a^T$ and is a contraction. As in the previous cases we can estimate each component separately.

Using the estimate (3.3.2) we obtain

$$\|D_x^s \Phi\|_{L_x^2} \leq \|\phi\|_s + c\|\partial_x \int_0^T U(t-t')D_x^s F(t')\|_{L_x^2} \leq \|D_x^s F\|_{L_x^1 L_T^2}. \tag{4.4.19}$$

Now, considering the term $3u_1^2 v_1^2 v_2$ in F and using Hölder's inequality, Leibniz rule and chain rule for fractional derivatives along with the estimate (4.2.9) we get

$$
\begin{aligned}
\|D_x^s(3u_1^2v_1^2v_2)\|_{L_x^1L_T^2} &\leq c\|D_x^s(u_1^2v_1^2)v_2\|_{L_x^1L_T^2} + c\|u_1^2v_1^2\|_{L_x^{\frac{5}{4}}L_T^{\frac{5}{2}}}\|D_x^s v_2\|_{L_x^5L_T^{10}} \\
&\leq c\|D_x^s(u_1^2v_1^2)\|_{L_x^{\frac{5}{4}}L_T^{\frac{5}{2}}}\|v_2\|_{L_x^5L_T^{10}} + c\|u_1v_1\|^2_{L_x^{\frac{5}{2}}L_T^5}\|D_x^s v_2\|_{L_x^5L_T^{10}} \\
&\leq c\big[\|D_x^s(u_1^2)v_1^2\|_{L_x^{\frac{5}{4}}L_T^{\frac{5}{2}}} + \|u_1^2\|_{L_x^{\frac{5}{2}}L_T^5}\|D_x^s v_1^2\|_{L_x^{\frac{5}{2}}L_T^5}\big]\|v_2\|_{L_x^5L_T^{10}} \\
&\quad + \|u_1\|^2_{L_x^5L_T^{10}}\|v_1\|^2_{L_x^5L_T^{10}}\|D_x^s v_2\|_{L_x^5L_T^{10}} \\
&\leq c\big[\|D_x^s(u_1^2)\|_{L_x^{\frac{5}{2}}L_T^5}\|v_1^2\|_{L_x^{\frac{5}{2}}L_T^5} + \|u_1\|^2_{L_x^5L_T^{10}}\|D_x^s v_1\|_{L_x^5L_T^{10}}\|v_1\|_{L_x^5L_T^{10}}\big]\|v_2\|_{L_x^5L_T^{10}} \\
&\quad + \|u_1\|^2_{L_x^5L_T^{10}}\|v_1\|^2_{L_x^5L_T^{10}}\|D_x^s v_2\|_{L_x^5L_T^{10}} \\
&\leq c\big[\|D_x^s u_1\|_{L_x^5L_T^{10}}\|u_1\|_{L_x^5L_T^{10}}\|v_1\|^2_{L_x^5L_T^{10}} + \|u_1\|^2_{L_x^5L_T^{10}}\|D_x^s v_1\|_{L_x^5L_T^{10}}\|v_1\|_{L_x^5L_T^{10}}\big]\|v_2\|_{L_x^5L_T^{10}} \\
&\quad + \|u_1\|^2_{L_x^5L_T^{10}}\|v_1\|^2_{L_x^5L_T^{10}}\|D_x^s v_2\|_{L_x^5L_T^{10}} \\
&\leq cT^{4s/3}\big[\|D_x^s u_1\|_{L_x^5L_T^{10}}\|D_t^{s/3}u_1\|_{L_x^5L_T^{10}}\|D_t^{s/3}v_1\|^2_{L_x^5L_T^{10}} \\
&\qquad + \|D_t^{s/3}u_1\|^2_{L_x^5L_T^{10}}\|D_x^s v_1\|_{L_x^5L_T^{10}}\|D_t^{s/3}v_1\|_{L_x^5L_T^{10}}\big]\|D_t^{s/3}v_2\|_{L_x^5L_T^{10}} \\
&\quad + cT^{4s/3}\|D_t^{s/3}u_1\|^2_{L_x^5L_T^{10}}\|D_t^{s/3}v_1\|^2_{L_x^5L_T^{10}}\|D_x^s v_2\|_{L_x^5L_T^{10}} \\
&\leq cT^{4s/3}\|\!|(u_1,v_1)|\!\|_s^4\|\!|(u_2,v_2)|\!\|_s.
\end{aligned}
$$

We can obtain similar estimates for the other terms in F too and get from (4.4.19)

$$
\begin{aligned}
\|D_x^s\Phi\|_{L_x^2} &\leq c\|\phi_2\|_s + cT^{4s/3}\big\{\|\!|(u_1,v_1)|\!\|_s^4 + \|\!|(u_1,v_1)|\!\|_s^3\|\!|(u_2,v_2)|\!\|_s + \|\!|(u_1,v_1)|\!\|_s^2\|\!|(u_2,v_2)|\!\|_s^2 \\
&\qquad + \|\!|(u_1,v_1)|\!\|_s\|\!|(u_2,v_2)|\!\|_s^3 + \|\!|(u_2,v_2)|\!\|_s^4\big\}\|\!|(u_2,v_2)|\!\|_s \\
&\leq c\|\phi_2\|_s + cT^{4s/3}\big\{\|\!|(u_1,v_1)|\!\|_1^{4s} + \|\!|(u_1,v_1)|\!\|_1^{3s}\|\!|(u_2,v_2)|\!\|_s + \|\!|(u_1,v_1)|\!\|_1^{2s}\|\!|(u_2,v_2)|\!\|_s^2 \\
&\qquad + \|\!|(u_1,v_1)|\!\|_1^{s}\|\!|(u_2,v_2)|\!\|_s^3 + \|\!|(u_2,v_2)|\!\|_s^4\big\}\|\!|(u_2,v_2)|\!\|_s.
\end{aligned} \tag{4.4.20}
$$

With the analogous argument, we can get estimates similar to (4.4.20) for the other norms involved in the definition of $\|\!|\cdot|\!\|_s$ and obtain

$$
\|\!|\Phi|\!\|_s \leq \|\phi_2\|_{H^s} + cT^{4s/3}f\big(\|\!|(u_1,v_1)|\!\|_1, \|\!|(u_2,v_2)|\!\|_s\big)\|\!|(u_2,v_2)|\!\|_s, \tag{4.4.21}
$$

where f is an appropriate polynomial in its arguments.

Also, one can derive similar estimates for $\|\Psi\|_s$ and combine those with (4.4.21) to have

$$\|(\Phi,\Psi)\|_s \leq \|(\phi_2,\psi_2)\|_{H^s\times H^s} + cT^{4s/3} f\big(\|(u_1,v_1)\|_1, \|(u_2,v_2)\|_s\big)\|(u_2,v_2)\|_s. \tag{4.4.22}$$

Now, choosing $a=\max\{\|(\phi_1,\psi_1)\|_{X^1}, \|(\phi_2,\psi_2)\|_{X^s}\}$ (which is $\|(\phi_1,\psi_1)\|_{X^1}$) and choosing T as in Theorem 4.6 we get $cT^{4s/3}a^4 < 1/2$. Since, $f\big(\|(u_1,v_1)\|_1, \|(u_2,v_2)\|_s\big) \leq ca^4$, $\Phi\times\Psi$ maps the ball $\mathcal{X}_a^T$ into $\mathcal{X}_a^T$.

We can prove that the mapping $\Phi\times\Psi$ is a contraction with a similar argument. This concludes the proof of the theorem. □

The following proposition provides the L^2 norm estimates for the solution (u_2,v_2).

Proposition 4.1. *Define* $\|(u_2,v_2)\|_0 = \max\{\|u_2\|_0, \|v_2\|_0\}$ *where,*

$$\|f\|_0 = \|f\|_{L_T^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_T^2} + \|f\|_{L_x^5 L_T^{10}}.$$

Let (u_1,v_1) *and* (u_2,v_2) *be solutions to the IVPs (4.4.4) and (4.4.6) with* $(\phi_1,\psi_1)\in X^1$ *and* $(\phi_2,\psi_2)\in X^s$ *respectively satisfying* $\|(\phi_1,\psi_1)\|_{X^1}\sim N^{1-s}$ *and* $\|(\phi_2,\psi_2)\|_X \sim N^{-s}$, $0<s<1$. *Then*

$$\|(u_2,v_2)\|_0 \sim N^{-s}. \tag{4.4.23}$$

Proof. The proof follows by using the equivalent integral equation (4.4.17), the linear estimates and the choice of T in Theorem 4.7. So, we omit the details. □

The following Proposition gives the estimates for the X^1 and X norms of the inhomogeneous part of the evolution of the high frequency part.

Proposition 4.2. *Let F and G be given by (4.4.7) and (4.4.8) with (u_1,v_1) and (u_2,v_2) solutions to the IVPs (4.4.4) and (4.4.6) respectively. Define,*

$$(z_1(t),z_2(t)) = \Big(-\int_0^t U(t-t')\partial_x F(t')\,dt',\ -\int_0^t U(t-t')\partial_x G(t')\,dt'\Big). \tag{4.4.24}$$

Let (ϕ_1,ψ_1) and (ϕ_2,ψ_2) satisfy the hypothesis of Proposition 4.1. If $0<s<1$, then

$$\sup_{t\in[0,T]} \|(z_1(t),z_2(t))\|_{X^1} \leq cN^{1-2s} \tag{4.4.25}$$

and

$$\|(z_1(t),z_2(t))\|_X \leq cN^{-s}. \tag{4.4.26}$$

Proof. We apply the estimate (3.3.2) to get

$$\|\partial_x z_1\|_{L^2} = \|\partial_x \int_0^t U(t-t')\partial_x F(t')\,dt'\|_{L^2} \le c\|\partial_x F\|_{L^1_x L^2_T}. \tag{4.4.27}$$

As in the proof of the Theorem 4.7, we consider the term $u_1^2 v_1^2 v_2$ and use argument as in [28] together with the choice of T and estimates in Theorem 4.6 and Proposition 4.1 to get

$$\begin{aligned}
\|\partial_x(u_1^2v_1^2v_2)\|_{L^1_xL^2_T} &\le \|u_1^2v_1^2v_{2x}\|_{L^1_xL^2_T} + \|u_1^2v_1v_{1x}v_2\|_{L^1_xL^2_T} + \|u_1u_{1x}v_1^2v_2\|_{L^1_xL^2_T}\\
&\le c\|u_1\|^2_{L^4_xL^\infty_T}\|v_1\|^2_{L^4_xL^\infty_T}\|v_{2x}\|_{L^\infty_xL^2_T} + c\|u_1\|^2_{L^5_xL^{10}_T}\|v_1\|_{L^5_xL^{10}_T}\|v_{1x}\|_{L^5_xL^{10}_T}\|v_2\|_{L^5_xL^{10}_T}\\
&\quad + c\|u_1\|_{L^5_xL^{10}_T}\|u_{1x}\|_{L^5_xL^{10}_T}\|v_1\|^2_{L^5_xL^{10}_T}\|v_2\|_{L^5_xL^{10}_T}\\
&\le c\|u_1\|^2_{L^4_xL^\infty_T}\|v_1\|^2_{L^4_xL^\infty_T}\|v_{2x}\|_{L^\infty_xL^2_T} + cT\|D_t^{1/3}u_1\|^2_{L^5_xL^{10}_T}\|D_t^{1/3}v_1\|_{L^5_xL^{10}_T}\|v_{1x}\|_{L^5_xL^{10}_T}\|v_2\|_{L^5_xL^{10}_T}\\
&\quad + cT\|D_t^{1/3}u_1\|_{L^5_xL^{10}_T}\|u_{1x}\|_{L^5_xL^{10}_T}\|D_t^{1/3}v_1\|^2_{L^5_xL^{10}_T}\|v_2\|_{L^5_xL^{10}_T}\\
&\le c|\!|\!|(u_1,v_1)|\!|\!|^4_{\frac14}|\!|\!|(u_2,v_2)|\!|\!|_0 + cT|\!|\!|(u_1,v_1)|\!|\!|^4_1|\!|\!|(u_2,v_2)|\!|\!|_0\\
&\le c|\!|\!|(u_1,v_1)|\!|\!|_1|\!|\!|(u_2,v_2)|\!|\!|_0 + cT|\!|\!|(u_1,v_1)|\!|\!|^4_1|\!|\!|(u_2,v_2)|\!|\!|_0\\
&\le cN^{1-s}N^{-s} + cN^{-3(1-s)}N^{4(1-s)}N^{-s}\\
&\le cN^{1-2s}.
\end{aligned} \tag{4.4.28}$$

We can obtain similar estimates for the other terms in F too.

Using an analogous argument we can get,

$$\|z_1\|_{L^2_x} \le cN^{-s}.$$

Finally, we can also obtain similar estimates for z_2 and that concludes the proof. □

Now, we are in position to provide proof of the global well-posedness result.

Proof of Theorem 4.5. Let $(u_0,v_0) \in X^s(\mathbb{R})$, $0<s<1$ such that $\|(u_0,v_0)\|_X < \|(S,S)\|_X$. Also, consider $N \gg 1$ be arbitrary but fixed to be determined later. Let us decompose the initial data as in (3.4.1) to

$$\begin{cases} u_0(x) = \phi_1(x) + \phi_2(x),\\ v_0(x) = \psi_1(x) + \psi_2(x). \end{cases} \tag{4.4.29}$$

Then we have,

$$\begin{cases} \|(\phi_1,\psi_1)\|_{X^\beta} \le cN^{\beta(1-s)}, \qquad 0<\beta\le 1 \\ \|(\phi_1,\psi_1)\|_{X} < \|(S,S)\|_X. \end{cases} \tag{4.4.30}$$

$$\|(\phi_2,\psi_2)\|_{X^\rho} \le cN^{\rho-s}, \quad 0<\rho\le s<1. \tag{4.4.31}$$

Consider the IVP (4.4.4) with initial data $(\phi_1,\psi_1)\in X^1$. From Theorem 4.4 there exists T satisfying

$$T\le c\|(\phi_1,\psi_1)\|_{X^1}^{-3}\sim N^{-3(1-s)}, \tag{4.4.32}$$

such that the IVP (4.4.4) has a unique solution (u_1,v_1) in the interval $[0,T]$. Moreover, from (3.4.8) we have

$$\sup_{t\in[0,T]}\|(u_1(t),v_1(t))\|_{X^1}\le cN^{1-s}. \tag{4.4.33}$$

Now, we consider the IVP (4.4.6) with initial data (ϕ_2,ψ_2). In Theorem 4.7 we found that the IVP (4.4.6) has a unique solution (u_2,v_2) defined in the same interval of existence of the solution (u_1,v_1), $[0,T]$ and is given by (4.1.19), i.e.

$$\begin{cases} u_2(t)=U(t)\phi_2+z_1(t) \\ v_2(t)=U(t)\psi_2+z_2(t). \end{cases} \tag{4.4.34}$$

where $z_1(t)$ and $z_2(t)$ are given by (4.4.24).

As mentioned earlier, $u=u_1+u_2$ and $v=v_1+v_2$ solve the IVP (1.3.67) in the time interval $[0,T]$.

Given $\tilde T>0$ arbitrary, we are interested in extending the solution (u,v) of the IVP (1.3.67) to the interval $[0,\tilde T]$. For this, we iterate the above process in each interval of size T unless covering the whole interval. Now, at the time $t=T$ we have,

$$\begin{cases} u(T)=u_1(T)+U(T)\phi_2+z_1(T) \\ v(T)=v_1(T)+U(T)\psi_2+z_2(T). \end{cases} \tag{4.4.35}$$

Now we decompose $(u(T), v(T))$ as,

$$\begin{cases} u(T) = \tilde{u}_1(T) + \tilde{u}_2(T) \\ v(T) = \tilde{v}_1(T) + \tilde{v}_2(T), \end{cases} \tag{4.4.36}$$

where,

$$\begin{cases} \tilde{u}_1(T) = u_1(T) + z_1(T), \qquad \tilde{u}_2(T) = U(T)\phi_2 \\ \tilde{v}_1(T) = v_1(T) + z_2(T), \qquad \tilde{v}_2(T) = U(T)\psi_2, \end{cases} \tag{4.4.37}$$

and evolve $(\tilde{u}_1(T), \tilde{v}_1(T))$ and $(\tilde{u}_2(T), \tilde{v}_2(T))$ according to the IVPs (4.4.4) and (4.4.6) respectively. Using previous procedure, to get solution to the IVP (1.3.67) in $[T, 2T]$ we must guarantee that $(\tilde{u}_1(T), \tilde{v}_1(T))$ and $(\tilde{u}_2(T), \tilde{v}_2(T))$ satisfy the respective conditions (4.4.30) and (4.4.31).

Since $U(t)$ is unitary in H^ρ, $(\tilde{u}_2(T), \tilde{v}_2(T))$ satisfies the same growth condition as that of (ϕ_2, ψ_2), i.e, $(\tilde{u}_2(T), \tilde{v}_2(T)) \in X^\rho$ and

$$\|(\tilde{u}_2(T), \tilde{v}_2(T))\|_{X^\rho} = \|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho - s}, \quad \rho \leq s.$$

Now, let us check how is the growth of the X^1-norm and the X-norm of $(\tilde{u}_1(T), \tilde{v}_1(T))$. Using the estimate (3.5.5) and Proposition 4.2 we get

$$\begin{aligned} \|(\tilde{u}_1(T), \tilde{v}_1(T))\|_{X^1} &\leq \|(u_1(T), v_1(T))\|_{X^1} + \|(z_1(T), z_2(T))\|_{X^1} \\ &\leq cN^{1-s} + cN^{1-2s}. \end{aligned} \tag{4.4.38}$$

On the other hand, using (4.4.35) and the conservation law (1.3.61) we obtain,

$$\begin{aligned} \|(\tilde{u}_1(T), \tilde{v}_1(T))\|_X &\leq \|(u(T), v(T)) - (U(T)\phi_2, U(T)\psi_2)\|_X \\ &\leq \|(u(T), v(T))\|_X + \|(\phi_2, \psi_2)\|_X \\ &\leq \|(u_0, v_0)\|_X + N^{-s}. \end{aligned} \tag{4.4.39}$$

Therefore, choosing N sufficiently large enough, we obtain $\|(\tilde{u}_1(T), \tilde{v}_1(T))\|_X < \|(S,S)\|_X$. Hence, the conditions (4.4.30) and (4.4.31) are satisfied for the first iteration.

To cover the interval $[0, \tilde{T}]$ we must iterate the above process $\tilde{T}/T$ times. As seen earlier, in each iteration, there will be a contribution of $\|(z_1, z_2)\|_{X^1}$ and $\|(z_1, z_2)\|_X$. From (3.5.10) we see that the total contribution of $\|(z_1, z_2)\|_{X^1}$ to cover $[0, \tilde{T}]$ is, $(\tilde{T}/T)N^{1-2s}$.

Thus the X^1-norm of (z_1, z_2) will grow uniformly as N^{1-s} on the interval $[0, \tilde{T}]$ if we have,

$$\frac{\tilde{T}}{T} N^{1-2s} < cN^{1-s}. \tag{4.4.40}$$

Now, using $T \sim N^{-3(1-s)}$ from (4.4.32) we see that (4.4.40) is equivalent to,

$$\tilde{T} N^{3-4s} < c. \tag{4.4.41}$$

Therefore, to guarantee (3.5.12) we must choose $N = N(\tilde{T})$ large, satisfying

$$N(\tilde{T}) = \tilde{T}^{\frac{1}{4s-3}},$$

with $4s - 3 > 0$, i.e. $s > 3/4$.

Let us show, with this choice the X-norm is also controlled by $\|(S,S)\|_X$. We know from (3.5.11) that, in each step there is a contribution of N^{-s}. Therefore, the total contribution to cover the interval $[0, \tilde{T}]$ is $(\tilde{T}/T)N^{-s}$. Now, with the choice of N we get,

$$\frac{\tilde{T}}{T} N^{-s} \leq c\tilde{T} N^{3(1-s)} N^{-s} \leq c,$$

and we are done as in [28].

Hence, we conclude that the IVP (1.3.57) has global solution whenever $s > 3/4$. $\square$

Chapter 5

Unique Continuation Property

In this chapter we present some recent results on the unique continuation property for the bi-dimensional versions of the KdV equation, viz, the Zakharov-Kuznetsov equation and the Kadomtsev-Petviashvili equation.

We note that, after Carleman [19] initiated studies of UCP based on the weighted estimates for the associated solutions, many authors improved and extended Carleman's method to address parabolic and hyperbolic operators (see [35] and [65]). A detailed description on the recent developments in this subject for the dispersive models has been presented in the introduction. So we are not going to repeat here.

The author in [71] generalized the method introduced in [16] to address a bi-dimensional (spatial) model and provided with an affirmative answer to the question posed above for the ZK model. Although, employing this method, one can deduce UCP for the linear problem almost immediately, the same is not so simple for the nonlinear problem and is quite involved. The symbol associated with the linear operator and the appropriate choice of the parameter play important role in the approach we used. In what follows we address UCP for the ZK equation.

5.1 Zakharov-Kuznetsov Equation

In this section, we focus on the Zakharov-Kuznetsov (ZK) equation. Let us consider the following initial value problem (IVP),

$$\begin{cases} u_t + (u_{xx} + u_{yy})_x + uu_x = 0, \quad (x, y) \in \mathbb{R}^2, \ t \in \mathbb{R} \\ u(x, y, 0) = u_0(x, y), \end{cases} \tag{5.1.1}$$

where $u = u(x, y, t)$ is a real valued function.

This two dimensional generalization of the KdV equation was obtained by Zakharov and Kuznetsov [89] to describe the propagation of nonlinear ion-acoustic waves in magnetized plasma. Several properties of this equation including existence and stability of solitary wave solutions have extensively been studied in the literature (see for eg. [6], [26], [77]).

As mentioned earlier, our concern is about the following question: If a sufficiently smooth real valued solution $u = u(x, y, t)$ to the IVP (5.1.1) is supported compactly on a certain time interval, is it true that $u \equiv 0$? In some sense, it is a weak version of the unique continuation property (see Definition 3).

The result we present here is from [71] and is in the same spirit to that of the KdV equation obtained in [16]. For this purpose, we derive some new estimates to address a bi-dimensional (spatial) model and provide an affirmative answer to the question posed above. More precisely, we prove the following result.

Theorem 5.1. *Let $u = u(x, y, t)$ be a smooth solution to the IVP (5.1.1) and $I = [-T, T]$ be a non trivial time interval. If for some $B > 0$*

$$\operatorname{supp} u(t) \subseteq [-B, B] \times [-B, B], \quad \forall\, t \in I,$$

then $u \equiv 0$.

As mentioned in the introduction, there are much stronger results of UCP for the KdV and mKdV equation. For example, the UCP result due to Zhang in [93] implies the results in [75] and [16] for the KdV equation. Zhang [93] used inverse scattering theory and Miura's transformation to get these results. In fact, he introduced some decay condition to the solution and exploited the fact that the KdV and mKdV equations are completely integrable. Recently, Kenig, Ponce and Vega [49] proved that if a sufficiently smooth solution u of the generalized KdV equation is supported in $(-\infty, b)$ or in (a, ∞) at two different instants of time then $u \equiv 0$. To get this result they used Carleman's type estimate and the result due to Saut and Scheurer [75]. The exponential decay property of the solution is essential in the argument employed in [49].

Remark 5.1. *The equation* (5.1.1) *is not integrable (see [77] and [78]) and also we do not know whether its solution has exponential decay property. So the methods in [93] and [49] cannot be applied to get much stronger results as mentioned above.*

5.1.1 Preliminary Estimates

This section is devoted to establish some preliminary estimates that will play fundamental role in our analysis. Let us begin with the following result.

Lemma 5.1. *Let $u = u(x, y, t)$ be a smooth solution to the IVP (5.1.1). If for some $B > 0$,*

$$\operatorname{supp} u(t) \subseteq \mathcal{B} := [-B, B] \times [-B, B],$$

then for all $\lambda = (\xi, \eta), \sigma = (\theta, \delta) \in \mathbb{R}^2$, we have,

$$|\widehat{u(t)}(\lambda + i\sigma)| \lesssim e^{c|\sigma|B}. \tag{5.1.2}$$

Where we have used $|(x, y)| = \max\{|x|, |y|\}$.

Proof. Using the Cauchy-Schwarz inequality and the conservation law (1.4.72) we have,

$$\begin{aligned}
|\widehat{u(t)}(\lambda + i\sigma)| &\leq \int_{\mathbb{R}^2} |e^{-i\mathbf{x}\cdot(\lambda+i\sigma)} u(t)(\mathbf{x})| \, d\mathbf{x} \\
&\leq \int_{\mathcal{B}} e^{\mathbf{x}\cdot\sigma} |u(t)(\mathbf{x})| \, d\mathbf{x} \\
&\leq \max_{\mathbf{x}\in\mathcal{B}} e^{\mathbf{x}\cdot\sigma} \int_{\mathcal{B}} |u(t)(\mathbf{x})| \, d\mathbf{x} \\
&\leq c \max_{-B\leq x,y\leq B} e^{x\theta+y\delta} \Big(\int_{\mathbb{R}^2} |u(t)(\mathbf{x})|^2 \, d\mathbf{x} \Big)^{1/2} \\
&\leq c e^{B(|\theta|+|\delta|)} \lesssim e^{c|\sigma|B}.
\end{aligned}$$

□

For $\lambda = (\xi, \eta)$ and $\lambda' = (\xi', \eta')$ define

$$u^*(\lambda) = \sup_{t \in I} |\widehat{u(t)}(\lambda)| \tag{5.1.3}$$

and

$$m(\lambda) = \sup_{\substack{|\xi'| \geq |\xi| \\ |\eta'| \geq |\eta|}} |u^*(\lambda')|. \tag{5.1.4}$$

Considering $u(0)$ sufficiently smooth and taking into account the well-posedness theory for the IVP (5.1.1) (see for example, Biagioni and Linares [6]), we have the following result.

Lemma 5.2. *Let $u \in C([-T, T]; H^s)$ be a sufficiently smooth solution to the IVP (5.1.1) with $\operatorname{supp} u(t) \subseteq \mathcal{B},\ t \in I$, then for some constant B_1, we have,*

$$m(\lambda) \lesssim \frac{B_1}{1 + |\lambda|^4}. \tag{5.1.5}$$

Proof. The Cauchy-Schwarz inequality and the conservation law (1.4.72) yield,

$$\int_{\mathbb{R}^2} |u(t)(\lambda)|\, d\lambda \leq |\mathcal{B}|^{1/2} \|u(t)\|_{L^2} \lesssim 1. \tag{5.1.6}$$

Now, using properties of the Fourier transform and (5.1.6) we get,

$$\|\widehat{u(t)}\|_{L^\infty} \leq c\|u(t)\|_{L^1} \lesssim 1. \tag{5.1.7}$$

Therefore,

$$|\widehat{u(t)}(\lambda)| \leq \|\widehat{u(t)}\|_{L^\infty} \lesssim 1, \tag{5.1.8}$$

and consequently,

$$\sup_t |\widehat{u(t)}(\lambda)| \leq c. \tag{5.1.9}$$

From the local well-posedness result (see [6]), we have,

$$\|D^s u(t)\|_{L^\infty_T L^2_{xy}} \leq c. \tag{5.1.10}$$

Next, using the Cauchy-Schwarz inequality and (5.1.10) one gets,

$$\int_{\mathbb{R}^2} |D^s u(t)(x,y)|\, dxdy \leq c\Big(\int_{\mathbb{R}^2} |D^s u(t)(x,y)|^2\, dxdy\Big)^{1/2} \leq c. \tag{5.1.11}$$

Since,

$$|\lambda|^s \widehat{u(t)}(\lambda) = \widehat{D^s u(t)}(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}^2} D^s u(t)(x,y) e^{-i(x\xi+y\eta)}\, dxdy,$$

the estimate (5.1.11) implies,

$$|\lambda|^s |\widehat{u(t)}(\lambda)| \leq c\int_{\mathbb{R}^2} |D^s u(t)(x,y)|\, dxdy \leq c_1. \tag{5.1.12}$$

Therefore,

$$|\widehat{u(t)}(\lambda)| \leq \frac{c_1}{|\lambda|^s}. \tag{5.1.13}$$

If we consider $s = 4$ (which is possible, because we have local well-posedness for the IVP (5.1.1) in H^1) and combine (5.1.9) and (5.1.13) we get,

$$\sup_t |\widehat{u(t)}(\lambda)| \leq \frac{B_1}{1+|\lambda|^4}. \tag{5.1.14}$$

If λ' is such that $|\xi'| \geq |\xi|$ and $|\eta'| \geq |\eta|$, then $\frac{1}{1+|\lambda|^4} \geq \frac{1}{1+|\lambda'|^4}$.

Hence,

$$m(\lambda) = \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} \sup_{t\in I} |\widehat{u(t)}(\lambda')| \leq \frac{B_1}{1+|\lambda'|^4} \leq \frac{B_1}{1+|\lambda|^4},$$

as required. $\square$

Now, using Lemma 5.2, we have the following result.

Proposition 5.1. *Let $u(t)$ be compactly supported and suppose that there exists $t \in I$ with $u(t) \neq 0$. Then there exists a number $c > 0$ such that for any large number $Q > 0$ there are arbitrary large $|\lambda|$-values such that,*

$$m(\lambda) > c(m * m)(\lambda) \tag{5.1.15}$$

and

$$m(\lambda) > e^{-\frac{|\lambda|}{Q}}. \tag{5.1.16}$$

Proof. The argument is similar to the one given in the proof of lemma in page 440 in [16], so we omit it. □

Using the definition of $m(\lambda)$ and Proposition 5.1 we choose $|\lambda|$ large (with $|\xi|$, $|\eta|$ large) and $t_1 \in I$ such that,

$$|\widehat{u(t_1)}(\lambda)| = u^*(\lambda) = m(\lambda) > c(m * m)(\lambda) + e^{-\frac{|\lambda|}{Q}}. \tag{5.1.17}$$

In what follows we prove some estimates regarding derivative of an entire function which are crucial in the proof of the main results of this chapter. First, let us recall a lemma whose proof is given in [16].

Lemma 5.3. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function which is bounded and integrable on the real axis and satisfies,*

$$|\phi(\xi + i\theta)| \lesssim e^{|\theta|B}, \qquad \xi,\, \theta \in \mathbb{R}.$$

Then, for $\lambda_1 \in \mathbb{R}^+$ we have,

$$|\phi'(\lambda_1)| \lesssim B\Big(\sup_{\xi' \geq \lambda_1} |\phi(\xi')|\Big)\Big[1 + \Big|\log\Big(\sup_{\xi' \geq \lambda_1} |\phi(\xi')|\Big)\Big|\Big]. \tag{5.1.18}$$

Using this lemma, we have the following result.

Lemma 5.4. *Let $\Phi : \mathbb{C}^2 \to \mathbb{C}$ be an entire function satisfying*

$$|\Phi(\lambda + i\sigma)| \lesssim e^{c|\sigma|B} \qquad \lambda,\, \sigma \in \mathbb{R}^2,$$

such that for z_2 fixed, $\Phi_1(z_1) := \Phi(z_1, z_2)$ and for z_1 fixed, $\Phi_2(z_1) := \Phi(z_1, z_2)$ are bounded and integrable on the real axis. Then for $\lambda_1,\ \lambda_2 \in \mathbb{R}^+$ we have,

$$|\nabla\Phi(\lambda_1, \lambda_2)| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi', \eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi', \eta')|\Big)\Big|\Big]. \tag{5.1.19}$$

Proof. Let $\lambda' = (\xi', \eta')$ and fix z_2 such that $\eta' \geq \lambda_2$. Applying the Lemma 5.3 for Φ_1 we obtain,

$$|\Phi_1'(\lambda_1)| \lesssim B\Big(\sup_{\xi'\geq\lambda_1} |\Phi_1(\xi')|\Big)\Big[1 + \Big|\log\Big(\sup_{\xi'\geq\lambda_1} |\Phi_1(\xi')|\Big)\Big|\Big]. \tag{5.1.20}$$

Now, let us fix z_1 such that $\xi' \geq \lambda_1$. Again applying the Lemma 5.3 for Φ_2 we get,

$$|\Phi_2'(\lambda_2)| \lesssim B\Big(\sup_{\eta'\geq\lambda_2} |\Phi_1(\eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\eta'\geq\lambda_2} |\Phi_2(\eta')|\Big)\Big|\Big]. \tag{5.1.21}$$

Since

$$\nabla\Phi(\lambda_1, \lambda_2) := (\Phi_1'(\lambda_1), \Phi_2'(\lambda_2)),$$

we obtain,

$$\begin{aligned}
|\nabla\Phi(\lambda_1, \lambda_2)| &\lesssim B\ \max\Big\{\Big(\sup_{\xi'\geq\lambda_1} |\Phi(\xi', \eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\xi'\geq\lambda_1} |\Phi(\xi', \eta')|\Big)\Big|\Big], \\
&\qquad \Big(\sup_{\eta'\geq\lambda_2} |\Phi(\xi', \eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\eta'\geq\lambda_2} |\Phi(\xi', \eta')|\Big)\Big|\Big]\Big\} \\
&\leq B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi', \eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi', \eta')|\Big)\Big|\Big],
\end{aligned}$$

as needed. □

Corollary 5.1. *Let $\sigma \in \mathbb{R}^2$ be such that,*

$$|\sigma| \leq B^{-1}\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1>0\\ \eta'\geq\lambda_2>0}} |\Phi(\xi', \eta')|\Big)\Big|\Big]^{-1}. \tag{5.1.22}$$

Then,

$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda' + i\sigma)| \leq 4 \sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda')| \tag{5.1.23}$$

and

$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\nabla\Phi(\lambda' + i\sigma)| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda')|\Big)\Big|\Big]. \tag{5.1.24}$$

Proof. The proof of (5.1.23) is immediate by using Corollary 2.9 in [16]. In fact, first fixing $\eta' \geq \lambda_2$ and then fixing $\xi' \geq \lambda_1$ we obtain,

$$\begin{aligned}\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi'+i\theta,\eta'+i\delta)| &\leq \sup_{\eta'\geq\lambda_2}\Big(2\sup_{\xi'\geq\lambda_1}|\Phi(\xi',\eta'+i\delta)|\Big)\\ &\leq 4\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|.\end{aligned} \tag{5.1.25}$$

To prove (5.1.24), we use the estimate (5.1.23) and Lemma 5.4. For this, let us define $\tilde{\Phi}(z) = \Phi(z+i\sigma)$, then $\tilde{\Phi}$ is an entire function and moreover we have,

$$|\tilde{\Phi}(z+i\sigma')| = |\Phi(\lambda+i(\sigma+\sigma'))| \lesssim e^{c_1|\sigma+\sigma'|B} \lesssim e^{c_1|\sigma'|B}.$$

Therefore, $\tilde{\Phi}$ satisfies the conditions of Lemma 5.4 and we get,

$$|\nabla\tilde{\Phi}(\lambda_1,\lambda_2)| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\tilde{\Phi}(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\tilde{\Phi}(\xi',\eta')|\Big)\Big|\Big].$$

for any $\lambda_1,\lambda_2\in\mathbb{R}^+$.

Hence, using the definition of $\tilde{\Phi}$ and (5.1.23) we obtain,

$$\begin{aligned}&|\nabla\Phi(\lambda_1+i\theta,\lambda_2+i\delta)|\\ &\quad\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi'+i\theta,\eta'+i\delta)|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi'+i\theta,\eta'+i\delta)|\Big)\Big|\Big]\\ &\quad\lesssim 4B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(4\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big|\Big]\\ &\quad\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[(1+\log 4)+(1+\log 4)\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big|\Big]\\ &\quad\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big|\Big].\end{aligned} \tag{5.1.26}$$

Therefore,

$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\nabla\Phi(\lambda'+i\sigma')| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big|\Big],$$

which concludes the proof. $\square$

Corollary 5.2. *Let $t \in I$, $\Phi(z) = \widehat{u(t)}(z)$, σ be as in Corollary 5.1 and $m(\lambda)$ be as in (5.1.4). Then, for $|\sigma'| \leq |\sigma|$ fixed, we have*

$$|\nabla\Phi(\lambda - \lambda' + i\sigma')| \lesssim B\left[m(\lambda) + m(\lambda - \lambda')\right]\left[1 + |\log m(\lambda)|\right]. \tag{5.1.27}$$

Proof. Let $\tilde{\Phi}(z) := \Phi(z + i\sigma')$, $z = (z_1, z_2) = (\xi + i\theta, \eta + i\delta)$. First, let us use (5.1.24) with $\sigma = 0$ and then use (5.1.23) to get, for $|\bar{\lambda_1}| = \min\{|\xi_1|, |\xi - \xi_1|\}$, $|\bar{\lambda_2}| = \min\{|\eta|, |\eta - \eta'|\}$ and $\bar{\lambda} = (\bar{\lambda_1}, \bar{\lambda_2})$,

$$\begin{aligned}
|\nabla\Phi(\lambda - \lambda' + i\sigma')| &\leq \sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\nabla\Phi(\lambda' + i\sigma')| = \sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\nabla\tilde{\Phi}(\lambda')| \\
&\lesssim B\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\tilde{\Phi}(\lambda')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\tilde{\Phi}(\lambda')|\Big)\Big|\Big] \\
&\lesssim B\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\Phi(\lambda')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\Phi(\lambda')|\Big)\Big|\Big] \\
&\lesssim B\, m(\bar{\lambda})[1 + |\log m(\bar{\lambda})|] \\
&\leq B\,[m(\lambda) + m(\lambda - \lambda')][1 + |\log m(\lambda)|],
\end{aligned} \tag{5.1.28}$$

which is the desired estimate. $\square$

5.1.2 Proof of the UCP Result for the ZK Equation

In this section we present the proof of the UCP result for the ZK model.

Proof of Theorem 5.1: Suppose that there exists $t \in I$ such that $u(t) \neq 0$. We will use the estimates derived in the previous section to arrive at a contradiction. For this we proceed as follows. Form Duhamel's principle, we have for $t_1, t_2 \in I$

$$u(t_2) = U(t_2 - t_1)u(t_1) - \frac{1}{2}\int_{t_1}^{t_2} U(t_2 - t')(u^2)_x(t')\, dt', \tag{5.1.29}$$

where $U(t)$ given by,

$$U(t)f(x, y) = \int_{\mathbb{R}^2} e^{i(t(\xi^3 + \xi\eta^2) + x\xi + y\eta)}\hat{f}(\xi, \eta)\, d\xi d\eta,$$

is the unitary group associated to the linear problem. Taking Fourier transform in the space variables we get from (5.1.29),

$$\widehat{u(t_2)}(\lambda) = e^{i(t_2 - t_1)(\xi^3 + \xi\eta^2)}\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_{t_1}^{t_2} e^{i(t_2 - t')(\xi^3 + \xi\eta^2)}\widehat{u^2(t')}(\lambda)\, dt'. \tag{5.1.30}$$

Let $t_2 - t_1 = \Delta t$, then from (5.1.30) we obtain,

$$\widehat{u(t_2)}(\lambda) = e^{i\Delta t(\xi^3+\xi\eta^2)}\Big[\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_{t_1}^{t_2} e^{i(t_1-t')(\xi^3+\xi\eta^2)}\widehat{u^2(t')}(\lambda)\,dt'\Big]. \tag{5.1.31}$$

Let us change variable in the integral in (5.1.31) by defining $s = t' - t_1$, to get,

$$\widehat{u(t_2)}(\lambda) = e^{i\Delta t(\xi^3+\xi\eta^2)}\Big[\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_0^{\Delta t} e^{-is(\xi^3+\xi\eta^2)}\widehat{u^2(s+t_1)}(\lambda)\,ds\Big]. \tag{5.1.32}$$

Since $u(t), t \in I$ has compact support, by Paley-Wiener theorem, $\widehat{u(t)}(\lambda)$ has analytic continuation in $\mathbb{C}^2$, and we have,

$$\begin{aligned}\widehat{u(t_2)}(\lambda+i\sigma) = e^{i\Delta t\{(\xi+i\theta)^3+(\xi+i\theta)(\eta+i\delta)^2\}}\Big[&\widehat{u(t_1)}(\lambda+i\sigma)\\ &- \frac{i(\xi+i\theta)}{2}\int_0^{\Delta t} e^{-is\{(\xi+i\theta)^3+(\xi+i\theta)(\eta+i\delta)^2\}}\widehat{u^2(s+t_1)}(\lambda+i\sigma)\,ds\Big].\end{aligned} \tag{5.1.33}$$

Since,

$$\begin{aligned}(\xi+i\theta)^3 + (\xi+i\theta)(\eta+i\delta)^2 = \xi^3 &- 3\xi\theta^2 - \xi\eta^2 - \xi\delta^2 - 2\eta\theta\delta\\ &+ i(3\xi^2\theta - \theta^3 + 2\xi\eta\delta + \theta\eta^2 - \theta\delta^2),\end{aligned}$$

the use of Lemma 5.1 in (5.1.33) yields,

$$\begin{aligned}&ce^{\Delta t(3\xi^2\theta-\theta^3+2\xi\eta\delta+\theta(\eta^2-\delta^2))} \geq\\ &\geq \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - \frac{|\xi+i\theta|}{2}\int_0^{\Delta t} e^{s(3\xi^2\theta-\theta^3+2\xi\eta\delta+\theta(\eta^2-\delta^2))}\big|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{5.1.34}$$

Let us take $|\lambda| = \max\{|\xi|, |\eta|\}$ very large with both $|\xi|$ and $|\eta|$ large such that

$$\xi\eta > 0. \tag{5.1.35}$$

Choose $\sigma = \sigma(\lambda)$ with $|\sigma| = \max\{|\theta|, |\delta|\} \approx 0$ such that,

$$\theta\Delta t < 0 \qquad \text{and} \qquad \delta\Delta t < 0. \tag{5.1.36}$$

Moreover, let us suppose the following conditions are satisfied

$$\frac{1}{|\xi|} \ll \begin{cases} |\theta| \\ |\delta| \end{cases} \qquad \text{and} \qquad \frac{1}{|\eta|} \ll \begin{cases} |\theta| \\ |\delta|. \end{cases} \tag{5.1.37}$$

With these choices, (5.1.34) can be written as

$$\begin{aligned} &ce^{\Delta t(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)} \gtrsim \\ &\quad \gtrsim \left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - |\xi| \int_0^{\Delta t} e^{s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)} \left|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\right| ds. \end{aligned} \tag{5.1.38}$$

Now, using (5.1.35) and (5.1.36) in (5.1.38) we obtain for $\Delta t > 0$,

$$\Big|\int_0^{\Delta t} e^{s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)} \widehat{u^2(s+t_1)}(\lambda+i\sigma)\, ds\Big| = \Big|\int_0^{\Delta t} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \widehat{u^2(s+t_1)}(\lambda+i\sigma)\, ds\Big|$$

and for $\Delta t < 0$, making change of variables, $s \leftrightarrow -s$,

$$\begin{aligned} &\Big|\int_0^{\Delta t} e^{s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)} \widehat{u^2(s+t_1)}(\lambda+i\sigma)\, ds\Big| \\ &\qquad = \Big|\int_0^{-\Delta t} e^{-s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)} \widehat{u^2(t_1-s)}(\lambda+i\sigma)\, ds\Big| \\ &\qquad = \Big|\int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \widehat{u^2(t_1-s)}(\lambda+i\sigma)\, ds\Big|. \end{aligned}$$

Therefore, in any case we have,

$$\begin{aligned} e^{-|\Delta t|(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \gtrsim& \left|\widehat{u(t_1)}(\lambda+i\sigma)\right| \\ &- |\xi| \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \left|\widehat{u^2(t_1\pm s)}(\lambda+i\sigma)\right| ds. \end{aligned} \tag{5.1.39}$$

In what follows we consider the case $\Delta t > 0$ (the analysis for $\Delta t < 0$ is similar). Since $e^{-x} < 1$ for $x > 0$, the estimate (5.1.39) can be written as,

$$e^{-(3\xi^2+\eta^2)|\theta\Delta t|} \gtrsim \left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - |\xi| \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \left|\widehat{u^2(t_1+s)}(\lambda+i\sigma)\right| ds.$$

Finally, we write this last estimate in the following manner,

$$\begin{aligned}
e^{-(3\xi^2+\eta^2)|\theta\Delta t|} &\gtrsim \big|\widehat{u(t_1)}(\lambda)\big| - |\xi| \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \big|\widehat{u^2(t_1+s)}(\lambda)\big|\, ds \\
&\quad - \big|\widehat{u(t_1)}(\lambda+i\sigma) - \widehat{u(t_1)}(\lambda)\big| \\
&\quad - |\xi| \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \big|\widehat{u^2(t_1+s)}(\lambda+i\sigma) - \widehat{u^2(t_1+s)}(\lambda)\big|\, ds \\
&:= I_1 - I_2 - I_3.
\end{aligned} \tag{5.1.40}$$

Now, our aim is to find appropriate estimates for I_1, I_2 and I_3 to get a contradiction in (5.1.40).

Let us estimate I_1: Use of (5.1.3) and (5.1.15) yields,

$$\begin{aligned}
&|\xi| \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \big|\widehat{u(t_1+s)}\big| * \big|\widehat{u(t_1+s)}\big|(\lambda)\, ds \\
&\qquad\qquad \le |\xi|(u^* * u^*)(\lambda) \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} ds \\
&\qquad\qquad \le |\xi|(m*m)(\lambda) \frac{1-e^{-|\Delta t|(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}}{3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2} \\
&\qquad\qquad \le \frac{|\xi|(m*m)(\lambda)}{2|\xi\eta\delta|} \\
&\qquad\qquad \lesssim \frac{m(\lambda)}{2|\eta\delta|}.
\end{aligned}$$

Therefore,

$$I_1 \gtrsim m(\lambda) - \frac{m(\lambda)}{2|\eta\delta|} \ge \frac{m(\lambda)}{2}. \tag{5.1.41}$$

Now, we estimate I_2: For this let us define $\Phi(z) = \widehat{u(t_1)}(z)$, $z = (z_1, z_2) \in \mathbb{C}^2$. Using (5.1.17) we get,

$$|\Phi(\lambda)| = |\widehat{u(t_1)}(\lambda)| = \sup_{\substack{|\xi'|\ge|\xi| \\ |\eta'|\ge|\eta|}} |\Phi(\lambda')| = m(\lambda). \tag{5.1.42}$$

Let us choose $|\sigma|$ satisfying

$$|\sigma| \lesssim B^{-1}\big[1 + |\log m(\lambda)|\big]^{-1}, \tag{5.1.43}$$

and use Corollary 5.1 to obtain,

$$\begin{aligned} I_2 &= \big|\widehat{u(t_1)}(\lambda + i\sigma) - \widehat{u(t_1)}(\lambda)\big| \\ &\lesssim |\sigma| \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} |\nabla\widehat{u(t_1)}(\lambda' + i\sigma)| \\ &\lesssim |\sigma| B\, m(\lambda)\big[1 + |\log m(\lambda)|\big] \\ &\lesssim m(\lambda) \\ &\lesssim \frac{1}{5} m(\lambda). \end{aligned} \tag{5.1.44}$$

Next, we estimate I_3: Using Proposition 5.1, Corollary 5.2 and taking $|\sigma|$ as in (5.1.43) we get,

$$\begin{aligned} &\big|\widehat{u^2(t_1+s)}(\lambda + i\sigma) - \widehat{u^2(t_1+s)}(\lambda)\big| \\ &= \Big| \int_{\mathbb{R}^2} \widehat{u(t_1+s)}(\lambda + i\sigma - \lambda')\widehat{u(t_1+s)}(\lambda')\, d\lambda' - \int_{\mathbb{R}^2} \widehat{u(t_1+s)}(\lambda - \lambda')\widehat{u(t_1+s)}(\lambda')\, d\lambda' \Big| \\ &\leq \int_{\mathbb{R}^2} \big|\widehat{u(t_1+s)}(\lambda - \lambda' + i\sigma) - \widehat{u(t_1+s)}(\lambda - \lambda')\big| \big|\widehat{u(t_1+s)}(\lambda')\big|\, d\lambda' \\ &\leq |\sigma| \int_{\mathbb{R}^2} \sup_{|\sigma'|\leq|\sigma|} |\nabla\widehat{u(t_1+s)}(\lambda - \lambda' + i\sigma')\big|\, m(\lambda')\, d\lambda' \\ &\leq \int_{\mathbb{R}^2} [m(\lambda) + m(\lambda - \lambda')]\, m(\lambda')\, d\lambda' \\ &\leq m(\lambda)c_2 + (m * m)(\lambda) \\ &\leq m(\lambda)(c_2 + c^{-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} I_3 &\leq |\xi| m(\lambda)(c_2 + c^{-1}) \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta| + 2|\xi\eta\delta| + |\theta|\eta^2)} ds \\ &= |\xi| m(\lambda)(c_2 + c^{-1}) \frac{1 - e^{-|\Delta t|(3\xi^2|\theta| + 2|\xi\eta\delta| + |\theta|\eta^2)}}{3\xi^2|\theta| + 2|\xi\eta\delta| + |\theta|\eta^2} \\ &\leq \frac{|\xi| m(\lambda)(c_2 + c^{-1})}{2|\xi\eta\delta|} \\ &\lesssim \frac{m(\lambda)}{|\eta\delta|} \\ &< \frac{m(\lambda)}{5}. \end{aligned} \tag{5.1.45}$$

Now using (5.1.41), (5.1.44) and (5.1.45) in (5.1.40) we get,

$$e^{-(3\xi^2+\eta^2)|\theta\Delta t|} \gtrsim \frac{m(\lambda)}{2} - \frac{m(\lambda)}{5} - \frac{m(\lambda)}{5} = \frac{1}{10}m(\lambda) \gtrsim e^{-\frac{|\lambda|}{Q}}. \tag{5.1.46}$$

Our choice in (5.1.37) gives, $|\xi\theta| \gg 1$ and $|\eta\theta| \gg 1$. Therefore,

$$\begin{aligned} e^{-(3\xi^2+\eta^2)|\theta||\Delta t|} &= e^{-(3|\xi||\xi\theta|+|\eta||\eta\theta|)|\Delta t|} \\ &\leq e^{-(|\xi|+|\eta|)|\Delta t|} \\ &\leq e^{-c|\lambda||\Delta t|}. \end{aligned} \tag{5.1.47}$$

Finally, using (5.1.47) in (5.1.46), we obtain,

$$e^{-c|\lambda||\Delta t|} \gtrsim e^{-\frac{|\lambda|}{Q}},$$

which is a contradiction for $|\lambda|$ large, if we choose Q large such that $\frac{1}{Q} < c|\Delta t|$. This completes the proof of the theorem. $\square$

5.2 Kadomtsev-Petviashvili Equation

In this section we consider the following initial value problem (IVP) associated with the Kadomtsev-Petviashvili (KP) equation,

$$\begin{gathered} (u_t + u_{xxx} + uu_x)_x = \alpha u_{yy}, \quad (x,y) \in \mathbb{R}^2,\ t \in \mathbb{R} \\ u(x,y,0) = \phi(x,y), \end{gathered} \tag{5.2.48}$$

where $u = u(x,y,t)$ is a real valued function and $\alpha = \pm 1$. This model was derived by Kadomtsev and Petviashvili [45] to describe the propagation of weakly nonlinear long waves on the surface of fluid, when the wave motion is essentially one-directional with weak transverse effects along y-axis. Equation (5.2.48) is known as KP-I or KP-II equation according as $\alpha = 1$ or $\alpha = -1$.

The UCP result obtained for the ZK model in the previous section motivated us to think for the similar result for the KP equation (see [72]). Unlike ZK model, there is singularity in the associated symbol of the linear KP operator. So, one needs to handle the analysis with utmost care. The structure of the associated symbol has also influenced a lot in the well-posedness results for the Cauchy problem for the KP equation. In this sense, the KP-II equation is much better than the KP-I equation, see for example [15], [38]–[43], [66], [84], [85] and [88]. The structure of the associated symbol has also affected our result on UCP for the KP equation. Here, we are able to handle only the KP-II equation by choosing appropriate parameters, see Remark 5.3 below. Therefore, from here onwards, we concentrate our work on KP-II equation (i.e., the IVP (5.2.48) with $\alpha = -1$) and obtain UCP for it. More precisely, using the scheme employed for the Zakharov-Kuznetsov equation we prove the following theorem.

Theorem 5.2. *Let $u \in C(\mathbb{R}; H^s(\mathbb{R}^2))$ be a solution to the IVP associated with the KP-II equation with $s > 0$ large enough. If there exists a non trivial time interval $I = [-T, T]$ such that for some $B > 0$*

$$\operatorname{supp} u(t) \subseteq [-B, B] \times [-B, B], \quad \forall\, t \in I,$$

then $u \equiv 0$.

Remark 5.2. *In the context of the KdV equation* (1.3.58)*, several types unique continuation properties exist in the recent literature, see for example [49], [47], [75] and [93]. Since KP equation is known to be a two-dimensional version of the KdV equation, we believe that the other forms of the UCP as mentioned in the above references could be proved for the KP equation too, but this needs to be done. As far as we know, our result in Theorem 5.2 is the first UCP for the KP type models.*

To prove this theorem, using the principle of Duhamel, we write the IVP associated with the KP-II equation in the equivalent integral equation form,

$$u(t) = S(t)\phi - \int_0^t S(t-t')(uu_x)(t')\, dt', \tag{5.2.49}$$

where $S(t)$ given by,

$$S(t)f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(t(\xi^3 - \frac{\eta^2}{\xi}) + x\xi + y\eta)}\hat{f}(\xi,\eta)\, d\xi d\eta, \tag{5.2.50}$$

is the unitary group which describes the solution to the linear problem

$$\begin{aligned} &(u_t + u_{xxx})_x + u_{yy} = 0, \\ &u(x, y, 0) = f(x, y). \end{aligned} \tag{5.2.51}$$

5.2.1 Basic Estimates

This section is devoted to establish some basic estimates that will play fundamental role in our analysis. These estimates are not new and can be found in [16] and the author's previous work in [71]. We will not give the details of the proofs rather we just sketch the idea of the proof. Let us begin with the following result.

Lemma 5.5. *Let* $u \in C([-T, T]; H^s(\mathbb{R}^2))$ *be a sufficiently smooth solution to the IVP (5.2.48). If for some* $B > 0$*,* $\operatorname{supp} u(t) \subseteq \mathcal{B} := [-B, B] \times [-B, B]$*, then for all* $\lambda = (\xi, \eta), \sigma = (\theta, \delta) \in \mathbb{R}^2$*, we have*

$$|\widehat{u(t)}(\lambda + i\sigma)| \lesssim e^{c|\sigma|B}. \tag{5.2.52}$$

Where we have used $|(x, y)| = \max\{|x|, |y|\}$.

Proof. The proof follows using the Cauchy-Schwarz inequality and the conservation law (1.4.70) with the argument similar to the one given in the proof of Lemma 5.1. □

For $\lambda = (\xi, \eta)$ and $\lambda' = (\xi', \eta')$ consider u^* and a as defined earlier, i.e.,

$$u^*(\lambda) = \sup_{t \in I} |\widehat{u(t)}(\lambda)|, \tag{5.2.53}$$

$$a(\lambda) = \sup_{\substack{|\xi'| \geq |\xi| \\ |\eta'| \geq |\eta|}} |u^*(\lambda')|. \tag{5.2.54}$$

Considering ϕ sufficiently smooth and taking in to account the well-posedness theory for the IVP (5.2.48) (see for example, [16]), we have the following result.

Lemma 5.6. *Let* $u \in C([-T, T]; H^s(\mathbb{R}^2))$ *be a sufficiently smooth solution to the IVP (5.2.48) with* $\operatorname{supp} u(t) \subseteq \mathcal{B}$*,* $t \in I$*, then for some constant* B_1*, we have,*

$$a(\lambda) \lesssim \frac{B_1}{1 + |\lambda|^4}. \tag{5.2.55}$$

Proof. The idea of proof is the same as the one we used in Lemma 5.2. The Cauchy-Schwarz inequality and the conservation law (1.4.70) yield

$$\int_{\mathbb{R}^2} |u(t)(\lambda)|\, d\lambda \leq |\mathcal{B}|^{1/2} \|u(t)\|_{L^2} \lesssim 1. \tag{5.2.56}$$

Now, using properties of the Fourier transform and (5.2.56) we get

$$|\widehat{u(t)}(\lambda)| \leq \|\widehat{u(t)}\|_{L^\infty} \leq c\|u(t)\|_{L^1} \lesssim 1. \tag{5.2.57}$$

Therefore,

$$\sup_t |\widehat{u(t)}(\lambda)| \leq c. \tag{5.2.58}$$

From the local well-posedness result (see [15]), we have

$$\|D^s u(t)\|_{L^\infty_T L^2_{xy}} \leq c. \tag{5.2.59}$$

Since, $\operatorname{supp} u(t) \subseteq \mathcal{B}$ and

$$|\lambda|^s \widehat{u(t)}(\lambda) = \widehat{D^s u(t)}(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} D^s u(t)(x,y) e^{-i(x\xi + y\eta)}\, dxdy,$$

using the Cauchy-Schwarz inequality and the estimate (5.2.59) we obtain

$$|\lambda|^s |\widehat{u(t)}(\lambda)| \leq c \int_{\mathbb{R}^2} |D^s u(t)(x,y)|\, dxdy \leq c\Big(\int_{\mathbb{R}^2} |D^s u(t)(x,y)|^2\, dxdy \Big)^{1/2} \leq c_1. \tag{5.2.60}$$

Therefore,

$$|\widehat{u(t)}(\lambda)| \leq \frac{c_1}{|\lambda|^s}. \tag{5.2.61}$$

If we consider $s = 4$ (which is possible, since we have local well-posedness for the IVP (5.2.48), for eg., in H^1) and combine (5.2.58) and (5.2.61) we get

$$\sup_t |\widehat{u(t)}(\lambda)| \leq \frac{B_1}{1 + |\lambda|^4}. \tag{5.2.62}$$

If λ' is such that $|\xi'| \geq |\xi|$ and $|\eta'| \geq |\eta|$, then $\frac{1}{1+|\lambda|^4} \geq \frac{1}{1+|\lambda'|^4}$.

Hence

$$a(\lambda) = \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} \sup_{t\in I} |\widehat{u(t)}(\lambda')| \leq \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} \frac{B_1}{1+|\lambda'|^4} \leq \frac{B_1}{1+|\lambda|^4},$$

as required. □

As in the ZK equation, we have the following similar result in the case of the KP-II equation too.

Proposition 5.2. *Let $u(t)$ be compactly supported and suppose that there exists $t \in I$ with $u(t) \neq 0$. Then there exists a number $c > 0$ such that for any large number $Q > 0$ there are arbitrary large $|\lambda|$-values such that*

$$a(\lambda) > c(a * a)(\lambda), \tag{5.2.63}$$

$$a(\lambda) > e^{-\frac{|\lambda|}{Q}}. \tag{5.2.64}$$

Proof. The main ingredient in the proof is the estimate (5.2.55) in Lemma 5.6. The argument is similar to the one given in the proof of lemma in page 440 in [16], so we omit it. □

Using the definition of $a(\lambda)$ and Proposition 5.2 we choose $|\lambda|$ large (with $|\xi|$, $|\eta|$ large) and $t_1 \in I$ such that

$$|\widehat{u(t_1)}(\lambda)| = u^*(\lambda) = a(\lambda) > c(a * a)(\lambda) + e^{-\frac{|\lambda|}{Q}}. \tag{5.2.65}$$

Now we are ready to provide the proof of the UCP result for the KP-II equation.

5.2.2 Proof of the UCP Result for the KP-II Equation

This section is devoted to supply proof of Theorem 5.2. Although the scheme of the proof is analogous to the one we employed to get similar result for the ZK equation in Section 5.1.2, one needs to overcome some additional technical difficulties arising from the structure of the Fourier symbol associated to the linear KP-II equation.

Proof of Theorem 5.2. If possible, suppose $u(t) \neq 0$ for some $t \in I$. Our aim is to get a contradiction by using the estimates derived in the previous sections.

Let $t_1, t_2 \in I$, with t_1 as in (5.2.65). Using Duhamel's principle, we have

$$u(t_2) = S(t_2 - t_1)u(t_1) - \frac{1}{2}\int_{t_1}^{t_2} S(t_2 - t')(u^2)_x(t')\,dt'. \tag{5.2.66}$$

Taking Fourier transform in the space variables with $\lambda = (\xi, \eta)$ we get

$$\widehat{u(t_2)}(\lambda) = e^{i(t_2-t_1)(\xi^3-\frac{\eta^2}{\xi})}\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_{t_1}^{t_2} e^{i(t_2-t')(\xi^3-\frac{\eta^2}{\xi})}\widehat{u^2(t')}(\lambda)\,dt'. \tag{5.2.67}$$

Let $t_2 - t_1 = \Delta t$ and then make a change of variables $s = t' - t_1$ to get

$$\widehat{u(t_2)}(\lambda) = e^{i\Delta t(\xi^3-\frac{\eta^2}{\xi})}\Big[\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_0^{\Delta t} e^{-is(\xi^3-\frac{\eta^2}{\xi})}\widehat{u^2(s+t_1)}(\lambda)\,ds\Big]. \tag{5.2.68}$$

Since $u(t), t \in I$ has compact support, by Paley-Wiener theorem, $\widehat{u(t)}(\lambda)$ has analytic continuation in $\mathbb{C}^2$, and we have for $\sigma = (\theta, \delta)$

$$\begin{aligned}\widehat{u(t_2)}(\lambda + i\sigma) &= e^{i\Delta t\{(\xi+i\theta)^3-\frac{(\eta+i\delta)^2}{\xi+i\theta}\}}\Big[\widehat{u(t_1)}(\lambda + i\sigma)\\ &\quad - \frac{i(\xi+i\theta)}{2}\int_0^{\Delta t} e^{-is\{(\xi+i\theta)^3-\frac{(\eta+i\delta)^2}{\xi+i\theta}\}}\widehat{u^2(s+t_1)}(\lambda + i\sigma)\,ds\Big].\end{aligned} \tag{5.2.69}$$

Since

$$\begin{aligned}(\xi+i\theta)^3 - \frac{(\eta+i\delta)^2}{\xi+i\theta} &= \xi^3 - 3\xi\theta^2 - \frac{1}{\xi^2+\theta^2}(\xi\eta^2 - \xi\delta^2 + 2\eta\theta\delta)\\ &\quad + i\{3\xi^2\theta - \theta^3 - \frac{1}{\xi^2+\theta^2}(2\xi\eta\delta - \theta\eta^2 + \theta\delta^2)\},\end{aligned}$$

using Lemma 5.1 we get from (5.2.69)

$$\begin{aligned}&ce^{\Delta t\{3\xi^2\theta-\theta^3-\frac{2\xi\eta\delta-\theta(\eta^2-\delta^2)}{\xi^2+\theta^2}\}}\\ &\geq \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - \frac{|\xi+i\theta|}{2}\int_0^{\Delta t} e^{s\{3\xi^2\theta-\theta^3-\frac{2\xi\eta\delta-\theta(\eta^2-\delta^2)}{\xi^2+\theta^2}\}}\big|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{5.2.70}$$

Let us take $|\lambda| = \max\{|\xi|, |\eta|\}$ very large such that

$$\xi\eta > 0 \qquad \text{and} \qquad |\xi| \sim |\eta|. \tag{5.2.71}$$

Also, let us choose $\sigma = \sigma(\lambda)$ with $|\sigma| = \max\{|\theta|, |\delta|\} \approx 0$ with

$$\theta\Delta t < 0 \qquad \text{and} \qquad \delta\Delta t > 0. \tag{5.2.72}$$

Moreover, let us suppose the following conditions are satisfied

$$\frac{1}{|\xi|} \ll |\theta|,\, |\delta| \quad \text{and} \quad \frac{1}{|\eta|} \ll |\theta|,\, |\delta|. \tag{5.2.73}$$

With these choices, (5.2.70) can be written as

$$\begin{aligned} e^{\Delta t\{3\xi^2\theta - \frac{2\xi\eta\delta-\theta\eta^2}{\xi^2+\theta^2}\}} &\gtrsim \left|\widehat{u(t_1)}(\lambda+i\sigma)\right| \\ &\quad - |\xi| \int_0^{\Delta t} e^{s\{3\xi^2\theta - \frac{2\xi\eta\delta-\theta\eta^2}{\xi^2+\theta^2}\}} \left|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\right| ds. \end{aligned} \tag{5.2.74}$$

Now, taking into consideration of (5.2.71) and (5.2.72), the estimate (5.2.74) yields

$$\begin{aligned} &e^{-|\Delta t|\{3\xi^2|\theta| + \frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}} \\ &\gtrsim \left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - |\xi| \int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta| + \frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}} \left|\widehat{u^2(t_1 \pm s)}(\lambda+i\sigma)\right| ds. \end{aligned} \tag{5.2.75}$$

Where "+" sign corresponds to $\Delta t > 0$ and "$-$" sign corresponds to $\Delta t < 0$. In what follows we consider the case $\Delta t > 0$ (the case $\Delta t < 0$ is similar). Since $e^{-x} < 1$ for $x > 0$, (5.2.75) can be written as

$$\begin{aligned} &e^{-\{3\xi^2 + \frac{\eta^2}{\xi^2+\theta^2}\}|\theta\Delta t|} \\ &\gtrsim \left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - |\xi| \int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta| + \frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}} \left|\widehat{u^2(t_1+s)}(\lambda+i\sigma)\right| ds. \end{aligned}$$

Finally, this last estimate can be written as

$$
\begin{aligned}
& e^{-\{3\xi^2+\frac{\eta^2}{\xi^2+\theta^2}\}|\theta\Delta t|} \\
& \gtrsim \left|\widehat{u(t_1)}(\lambda)\right| - |\xi| \int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}} \left|\widehat{u^2(t_1+s)}(\lambda)\right| ds \\
& - \left|\widehat{u(t_1)}(\lambda+i\sigma) - \widehat{u(t_1)}(\lambda)\right| \\
& - |\xi| \int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}} \left|\widehat{u^2(t_1+s)}(\lambda+i\sigma) - \widehat{u^2(t_1+s)}(\lambda)\right| ds \\
& := I_1 - I_2 - I_3.
\end{aligned}
\tag{5.2.76}
$$

Now, we proceed to obtain appropriate estimates for I_1, I_2 and I_3 to arrive at a contradiction in (5.2.76). To obtain estimate for I_1, we use definition of $u^*(\lambda)$, i.e., (5.2.53) and the estimate (5.2.63) to get

$$
\begin{aligned}
& |\xi| \int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}} \left|\widehat{u(t_1+s)}\right| * \left|\widehat{u(t_1+s)}\right|(\lambda)\, ds \\
& \leq |\xi|(u^* * u^*)(\lambda) \int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}} ds \\
& \leq |\xi|(a * a)(\lambda) \frac{1 - e^{-|\Delta t|\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}}{3\xi^2|\theta| + \frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}} \\
& \leq \frac{|\xi|(a * a)(\lambda)}{3|\xi||\xi\theta|} \lesssim \frac{a(\lambda)}{3|\xi\theta|}.
\end{aligned}
$$

Therefore,

$$
I_1 \gtrsim a(\lambda) - \frac{a(\lambda)}{3|\xi\theta|} \geq \frac{a(\lambda)}{3}. \tag{5.2.77}
$$

To get estimate for I_2, let us define $\Phi(z) = \widehat{u(t_1)}(z)$, $z = (z_1, z_2) \in \mathbb{C}^2$. Now, using (5.1.17) one can obtain

$$
|\Phi(\lambda)| = |\widehat{u(t_1)}(\lambda)| = \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} |\Phi(\lambda')| = a(\lambda). \tag{5.2.78}
$$

Let us choose $|\sigma|$ satisfying

$$|\sigma| \lesssim B^{-1}\big[1 + |\log a(\lambda)|\big]^{-1}, \tag{5.2.79}$$

and use Corollary 5.1 to obtain

$$I_2 \lesssim |\sigma| \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} |\nabla \widehat{u(t_1)}(\lambda' + i\sigma)| \lesssim |\sigma| B\, a(\lambda)\big[1 + |\log a(\lambda)|\big] \lesssim a(\lambda) \lesssim \frac{1}{15} a(\lambda). \tag{5.2.80}$$

To obtain estimate for I_3, we use Proposition 5.2, Corollary 5.2 and $|\sigma|$ as in (5.2.79) to get

$$\begin{aligned}
&\big|\widehat{u^2(t_1+s)}(\lambda + i\sigma) - \widehat{u^2(t_1+s)}(\lambda)\big| \\
&\leq \int_{\mathbb{R}^2} \big|\widehat{u(t_1+s)}(\lambda - \lambda' + i\sigma) - \widehat{u(t_1+s)}(\lambda - \lambda')\big|\big|\widehat{u(t_1+s)}(\lambda')\big|\, d\lambda' \\
&\leq |\sigma| \int_{\mathbb{R}^2} \sup_{|\sigma'|\leq|\sigma|} |\nabla \widehat{u(t_1+s)}(\lambda - \lambda' + i\sigma')\big|\, a(\lambda')\, d\lambda' \\
&\leq \int_{\mathbb{R}^2} [a(\lambda) + a(\lambda - \lambda')]\, a(\lambda')\, d\lambda' \\
&\leq a(\lambda) c_2 + (a * a)(\lambda) \\
&\leq a(\lambda)(c_2 + c^{-1}) \lesssim a(\lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_3 &\lesssim |\xi| a(\lambda) \int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta| + \frac{2|\xi\eta\delta| + |\theta|\eta^2}{\xi^2+\theta^2}\}} ds \\
&= |\xi| a(\lambda)\, \frac{1 - e^{-|\Delta t|\{3\xi^2|\theta| + \frac{2|\xi\eta\delta| + |\theta|\eta^2}{\xi^2+\theta^2}\}}}{3\xi^2|\theta| + \frac{2|\xi\eta\delta| + |\theta|\eta^2}{\xi^2+\theta^2}} \\
&\leq \frac{|\xi| a(\lambda)}{3|\xi^2\theta|} \\
&\lesssim \frac{a(\lambda)}{|\xi\theta|} < \frac{a(\lambda)}{15}.
\end{aligned} \tag{5.2.81}$$

Now inserting (5.2.77), (5.2.80) and (5.2.81) in (5.2.76) and using the estimate (5.2.64) we get

$$e^{-\{3\xi^2 + \frac{\eta^2}{\xi^2+\theta^2}\}|\theta\Delta t|} \gtrsim \frac{a(\lambda)}{3} - \frac{a(\lambda)}{15} - \frac{a(\lambda)}{15} = \frac{1}{3} a(\lambda) \gtrsim e^{-\frac{|\lambda|}{Q}}. \tag{5.2.82}$$

On the other hand, using (5.2.71) and (5.2.73) one can easily deduce

$$e^{-\{3\xi^2+\frac{\eta^2}{\xi^2+\theta^2}\}|\theta||\Delta t|} \leq e^{-|\lambda||\Delta t|}. \tag{5.2.83}$$

Hence, using (5.2.83) in (5.1.46), we arrive at

$$e^{-|\lambda||\Delta t|} \gtrsim e^{-\frac{|\lambda|}{Q}},$$

which is a contradiction for $|\lambda|$ large, if we choose Q large enough such that $\frac{1}{Q} < |\Delta t|$. $\square$

Remark 5.3. *Note that the Fourier symbol associated with the linear KP-I operator is* $\xi^3 + \frac{\eta^2}{\xi}$. *In this case we cannot make choice as in* (5.2.71) *and* (5.2.72) *to obtain estimate like in* (5.2.75) *with term of the form* $e^{-|\Delta t|\gamma}$, *for some* $\gamma > 0$. *As seen in the proof of Theorem 5.2, existence of such term in the RHS of* (5.2.75) *is very essential in the argument we employed. It would be interesting to obtain UCP for the KP-I equation employing some other argument or modification.*

5.3 Comments

As discussed earlier, the unique continuation property (UCP) results that we presented here are, in some sense, in the weak form. So, it would be interesting to prove the strong UCP for the bi-dimensional dispersive models. We note that there are some improvements in this direction due to Isaja et. al. in [39, 40] for the KP-II and the ZK equations. The results in [39, 40] depend on the UCP results presented in this chapter.

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